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An improved information volume of mass function based on plausibility transformation method

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ABSTRACT

The information volume of mass function (IVMF) is an effective tool for measuring the uncertainty of basic probability assignments in power sets. However, the current IVMF will yield counterintuitive results when applied to an inconsistent frame of discernment (FOD). To address this issue, an improved IVMF based on plausibility transformation method (PTM) is proposed in this paper. Compared to existing methods, the proposed method yields a more reasonable result in cases where the FOD is inconsistent. Additionally, the proposed IVMF can be viewed as a geometric mean of first-order information volume and higher-order information volume, which can degenerate into Shannon Entropy in a probability distribution. The efficacy and rationality of the proposed IVMF are demonstrated through a series of numerical examples and an application in threat assessment.

1. Introduction

Shannon entropy (Shannon, 1948) is a fundamental concept in information theory. Given a probability distribution, Shannon entropy can measure the uncertainty of information. From the study of communication to the analysis of neural networks and genetics, information theory has found a vast array of applications, making it a foundational concept in modern science.

Though probability theory has long been the dominant approach to reasoning under uncertainty, there are some situations in real life where it is difficult to express the uncertainty in probability theory (Che et al., 2022). To address this issue, many alternatives have been presented in recent decades, including Dempster–Shafer evidence theory (Dempster, 2008; Shafer, 1976), fuzzy set (Van Laarhoven & Pedrycz, 1983; Zadeh, 1965), evidential reasoning (Zhou et al., 2023; Zhou, Zhou et al., 2022), Z numbers (Jiang, Cao et al., 2019; Liu et al., 2019; Luo & Deng, 2020), D numbers (Deng & Jiang, 2019; Liu & Deng, 2019; Liu & Zhang, 2020), soft sets (Alcantud et al., 2019; Feng et al., 2020), rough sets (Fujita et al., 2019; Pawlak, 1982), and PRS (Deng, 2022). Because Dempster–Shafer evidence theory is effective to handle uncertainty, it has extended to complex domain (Xiao, 2021a, 2021c, 2022a; Xiao, Cao et al., 2022) and quantum theory (Xiao, 2022b; Xiao & Pedrycz, 2022; Zhou, Tian et al., 2023). It also brings various applications in different fields, such as unclear information description (Xiao et al., 2020), decision-making in emergency (Fei & Ma, 2023; Fei & Wang, 2022a, 2022b), knowledge management (Anjaria, 2022), portfolio construction (Bisht & Kumar, 2023), link prediction (Fang et al., 2022), cellular automaton (Kharazmi & Contreras-Reyes, 2023) and so on.

However, it also comes with new problems, such as determining the extent of uncertainty under the framework of discernment (Deng, 2020b) and combining evidence in conflict (Abellán et al., 2021). To address the former issue, various methods have been proposed, including JS entropy (Jiroušek & Shenoy, 2018), SU measurement (Wang & Song, 2018), FB entropy (Zhou & Deng, 2022a), time fractal-based entropy (Zhou & Deng, 2022b) and other modes of analysis (Cui & Deng, 2023; Cui et al., 2022; Dutta & Shome, 2022; Zhou, Zhu et al., 2022). Among those, the belief entropy, also named Deng entropy (Deng, 2016), made it a natural fit for use in evidence theory (Abellán, 2017). In recent years, Deng entropy has been applied in a variety of fields, such as pattern recognition (Cui et al., 2019; Kazemi et al., 2021), data fusion (Richter et al., 2022; Tang et al., 2018), decision-making (Pan & Gao, 2023), as well as the field of complex numbers (Pan & Deng, 2023). Based on the splitting method to divide mass functions, the information volume of mass function (IVMF) (Deng, 2020a) is proposed using Deng entropy, which has also gained acceptance and been applied in many fields (Deng & Deng, 2021; Zhou & Deng, 2022a, 2022b).

Abbreviations: IVMF, information volume of mass function; PTM, plausibility transformation method; PPT, pignistic probability transformation

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However, the existing information volume has some open issues. For example, when the frame of discernment of two different mass functions does not agree with each other, i.e. although the elements in the mass function with consistent cardinality differ, they have the same BPA distribution. Then the existing IVMF will not be able to distinguish them and will give the same result, which is not reasonable.

To address this issue, an improved IVMF based on plausibility transformation method (PTM) is proposed in this paper. Compared with the existing method, the proposed IVMF has the following advantages:

- The proposed IVMF constitutes a geometric mean of first-order IVMF and higher-order IVMF.
- The proposed method corresponds to Shannon entropy if the mass functions degenerate to a probability distribution.
- Compared with the existing IVMF, the proposed method obtains a more reasonable result when the FOD of given mass functions is inconsistent.

In real-world applications such as expert systems, the majority of the decision-making process involves uncertain information processing, which can be modeled by evidence theory. However, most of the previous applications of evidence theory did not consider the situations of measuring uncertainty of a certain period instead of a specific moment in time. Inspired by Zhou and Deng (2022b)' s explanation about the physical meaning of Deng entropy, splitting-based uncertainty measures like IVMF can be used for measuring the information volume of a certain period, while other kinds of uncertainty methods are not applicable here. This paper applies IVMF to the application area of threat assessment for the first time. Compared with the existing IVMF, The proposed method can better handle the case when the frame of discernment is inconsistent due to insufficient information, equipment malfunction, or other constraints.

This paper is structured as follows. Section 2 briefly introduces some pertinent concepts. Section 3 presents the proposed IVMF, followed by some numerical examples in Section 4. Section 5 provides an application to validate the proposed method with a comparison to the existing methods. Finally, Section 6 concludes this paper.

2. Preliminaries

2.1. Dempster-Shafer evidence theory

How to model and measure the uncertainty and dynamics of the system (Chu et al., 2022; Wang, Hou et al., 2022; Wang, Mu et al., 2022; Wang et al., 2023) have attracted a lot of attention, and the Dempster–Shafer evidence theory is one of the alternative solutions. In this context, some preliminary concepts are introduced in this subsection.

Definition 1 (*BPA Dempster, 2008*). An n-element finite set Θ denoted $\Theta = \{t_1, t_2, \dots, t_n\}$, is called the frame of discernment (FOD).

A basic probability assignment (BPA), or the mass function, constitutes a mapping from the power set of Θ marked as 2^{Θ} to the interval $[0,1]$. For any element A_i in 2^{Θ}, the mass function $m(A_i)$ satisfies:

$$m(\emptyset) = 0; \quad \sum_{A_i \in 2^{\Theta}} m\left(A_i\right) = 1; \quad m\left(A_i\right) \geqslant 0.$$

Definition 2 (*Belief Function Shafer, 1976*). For a FOD Θ with n elements, and its BPA $\mathbb{B}\left(2^{\Theta}\right)$, the belief (Bel) function and the plausibility (Pl) function are defined as follows.

$$Bel\left(A_i\right) = \sum_{B_i \subseteq A_i} m\left(B_i\right) = 1 - Pl\left(\overline{A_i}\right), \tag{1}$$

$$Pl\left(A_i\right) = \sum_{B_i \cap A_i \neq \emptyset \wedge B_i \subseteq \Theta} m\left(B_i\right) = 1 - Bel\left(\overline{A_i}\right). \tag{2}$$

2.2. Shannon entropy

Entropy is significant when it comes to measuring uncertainty in information theory, which is developed and applied in many fields, such as divergence (Xiao, Wen et al., 2022), information quality (Xiao, 2021b), negation (Xiao, 2021c; Xiao & Pedrycz, 2022), etc. Especially, Shannon entropy has been applied not only in information theory but also in probability theory.

Definition 3 (*Shannon Entropy Shannon, 1948*). For a probability distribution $P = \{p_1, p_2, \dots, p_n\}$, the Shannon entropy of P is defined as

$$E_S(P) = -\sum_{p_i \in P} p_i \log(p_i), \tag{3}$$

where p_i satisfies $\sum_{i=1}^{n} p_i = 1$.

2.3. Deng entropy and information volume of mass function

Definition 4 (*Deng Entropy Deng, 2016*). Given a FOD Θ and its corresponding BPA, Deng entropy is defined as

$$E_D(m) = -\sum_{A_i \subseteq \Theta} m(A_i) \log \frac{m\left(A_i\right)}{2^{|A_i|} - 1}, \tag{4}$$

where $|A_i|$ is the cardinality of A.

Deng entropy can degenerate into Shannon entropy when the mass function is defined for individual elements. When splitting the mass function into its power set, more information within the mass function is considered. Thus, Deng proposed the IVMF (Deng, 2020a) based on Deng entropy.

Definition 5 (*Information Volume of Mass Function Deng, 2020a*). For an n-element FOD Θ, the information volume of mass function (IVMF) is defined as

$$H_{IV} = \lim_{n \to \infty} \left(\sum_{t=0}^{n} \sum_{A_i \in \Theta} m^{(t)}(A_i) \log \frac{m\left(A_i\right)}{2^{|A_i|} - 1} + \sum_{B_i \subseteq \Theta \wedge |B_i| > 1} m^{(n)}(B_i) \right), \tag{5}$$

where $m^{(n)}(A_i)$ is the corresponding nth time of splitting the mass function of A_i based on the distribution of maximum Deng entropy (Kang & Deng, 2019):

$$m\left(F_i\right) = \frac{\left(2^{|F_i|} - 1\right)}{\sum_{G_i \subseteq F_i} \left(2^{|G_i|} - 1\right)}. \tag{6}$$

To better understand the splitting process in calculation, Fig. 1 shows the splitting method of a vacuous BPA: $\mathbb{B}(\Theta) = \{m(\{\Theta\}) = m(\{\theta_1\}) = m(\{\theta_2\}) = \frac{1}{3}\}$, where $\Theta = \{\theta_1, \theta_2\}$.

2.4. Some probability transformation methods

Definition 6 (*Plausibility Transformation Method Cobb & Shenoy, 2006*). For an n-element FOD Θ with its BPA $\mathbb{B}(2^{\Theta})$, the plausibility transformation method (PTM) is defined as

$$Pl_m\left(\theta_i\right) = \frac{Pl\left(\theta_i\right)}{\sum_{j=1}^{n} Pl\left(\theta_j\right)}. \tag{7}$$

Definition 7 (*Pignistic Probability Transformation Smets, 2005*). The pignistic probability transformation (PPT) of an n-element FOD Θ with its BPA $\mathbb{B}(2^{\Theta})$ is defined as

$$BetP\left(\theta_i\right) = \sum_{\theta_i \in A_i \wedge A_i \in 2^{\Theta}} \frac{m(A_i)}{|A_i|}. \tag{8}$$

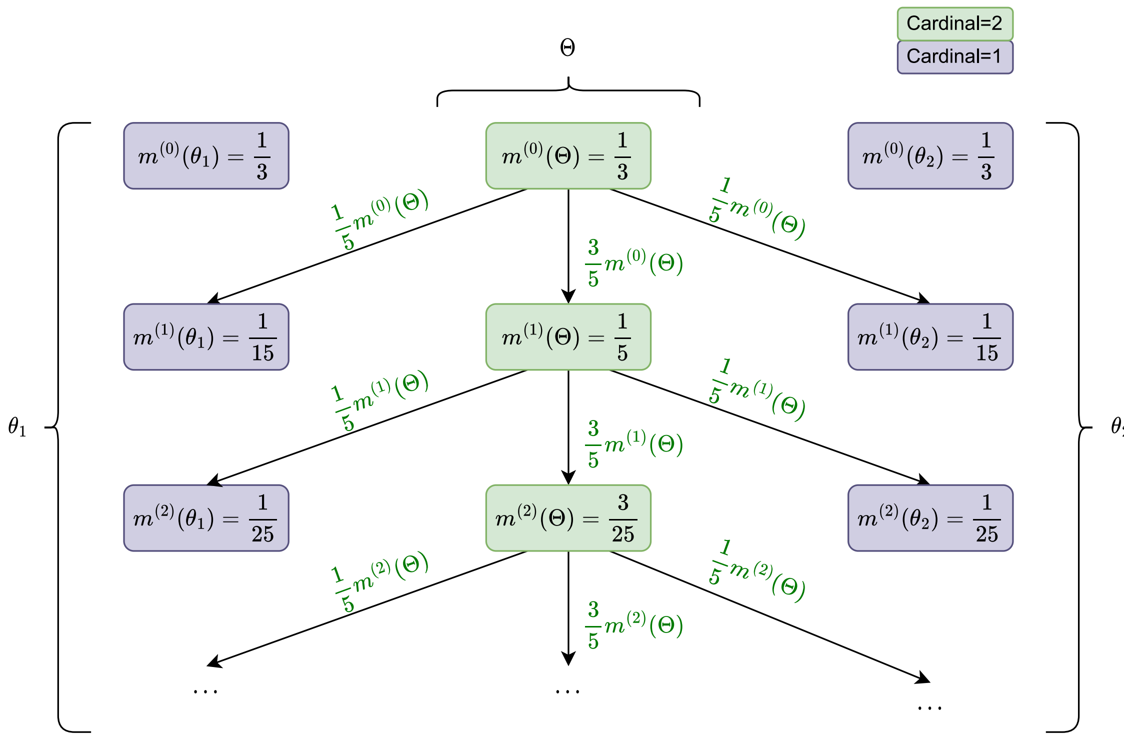


Fig. 1. The splitting process of IVMF in Definition 5.

Though PTM does not satisfy upper and lower bounds consistency, it is one of the probability transformation methods consistent with Dempster's combination rule (Han et al., 2016). As for PPT, it can be seen as redistributing the mass function of compound focal elements to singletons included in them equally, which may result in some counterintuitive results. More details are given in Section 4.

Apart from PPT and PTM which are based on probability transformation, there are other kinds of probability transformation methods based on different aspects, like uncertainty minimization (Pan & Deng, 2020), belief interval (Deng & Wang, 2020), evidential correlation (Jiang, Huang et al., 2019), ordered weight network (Chen et al., 2021) and importance weight (Zhao et al., 2023).

In Section 3, the definition of the improved IVMF will be given, which is based on PTM.

2.5. Uncertainty measures

In evidence theory, various uncertainty measures are proposed by many scholars. Some measures and entropy used later in comparison with the proposed method are listed in Table 1.

3. The proposed method

Given a BPA, the information volume can be measured by Deng's method (Deng, 2020a). However, when the FOD of two different BPAs does not agree with each other, the result of Deng's IVMF gives counterintuitive results. Hence, the paper proposed a new kind of IVMF to solve this problem. In Section 3, the proposed IVMF is presented based on the previous work of Deng's IVMF (Deng, 2020a) and PTM (Cobb & Shenoy, 2006). The reasons for choosing PTM instead of PPT will be discussed in Sections 4 and 5.

Definition 8 (*Improved Information Volume of Mass Function*)**.** Given a FOD $\Theta = \{\theta_1, \theta_2 \dots, \theta_n\}$ with its BPA: $\mathbb{B}(2^{\Theta}) : m(A_i)\ for\ A_i \in 2^{\Theta}$. The proposed IVMF is defined as follows.

$$H_{IV'-PTM} = \sqrt{H_{IV} \cdot E_S(Pl_m(\theta_i))} = \sqrt{H_{IV} \cdot \sum_{i=1}^{n} Pl_m(\theta_i) \log Pl_m(\theta_i)}. \tag{9}$$

where H_{IV} is Deng's IVMF (Deng & Deng, 2021), while E_S and the $Pl_m(\theta_i)$ refer to the Shannon entropy and PTM, respectively.

In Section 4, numerical examples are given to demonstrate the effectiveness of the proposed method, including a comparison to Deng's IVMF.

4. Examples and discussion

Example 1. Supposed a FOD: $\Theta = \{t_1, t_2\}$, there are two BPAs defined as:

$$m_1(\{t_1\}) = 4/7, \quad m_1(\{t_2\}) = 3/7,$$
$$m_2(\{t_1\}) = 1/4, \quad m_2(\{\Theta\}) = 3/4.$$

Example 1 is used to illustrate the calculation process. For m_1, and based on Definition 5, the result of IVMF denoted as H_{IV} is

$$H_{IV} = -4/7 \log(4/7) - 3/7 \log(3/7) \approx 0.985228. \tag{10}$$

The corresponding mass functions of PTM are equal to the initial mass functions since there is only a singleton element in the mass function. Then the sum of Shannon entropy can be expressed as

$$\sum_{i=1}^{2} E_S(Pl_m(t_i)) = -4/7 \log(4/7) - 3/7 \log(3/7) = H_{IV} \approx 0.985228, \tag{11}$$

where $\quad Pl_m(t_i) = m_1(t_i) \quad i = 1, 2.$

Based on Eqs. (10) and (11) and Definition 8, the result of the proposed IVMF marked as $H_{IV'-PTM}$ is

$$H_{IV'-PTM} = \sqrt{H_{IV} \cdot E_S(Pl_m(t_i))} \approx 0.985228.$$

As for m_2, the H_{IV} is calculated based on Definition 5:

$$H_{IV} = \lim_{n\to\infty} \left(\sum_{t=0}^{n} \sum_{A_i \in \Theta} m^{(t)}(A_i) \log \frac{m(A_i)}{2^{|A_i|} - 1} + \sum_{B_i \subseteq \Theta \wedge |B_i| > 1} m^{(n)}(B_i) \right) \tag{12}$$
$$\approx 3.381670.$$

Table 1
Uncertainty measures listed above are used to compare with the proposed method.

Method	Expression	
	Maximum distribution	Maximum value
Hohle's measure	$C(m) = -\sum_{A_i \in 2^\Theta} m(A_i) \log Bel(A_i)$	
(Höhle, 1982)	$m(\theta_i) = \frac{1}{\lvert\Theta\rvert}$	$\log(\lvert\Theta\rvert)$
Yager's measure	$E(m) = -\sum_{A_i \in 2^\Theta} m(A_i) \log Pl(A_i)$	
(Yager, 2008)	$m(A_i) = \frac{1}{K}, \forall 1 \to K$ $\{F_1\} \cap \cdots \cap \{F_K\} = \varnothing$	$\log(\lvert\Theta\rvert)$
Weighted Hartley entropy	$E_{DP}(m) = -\sum_{A_i \in 2^\Theta} m(A_i) \log \lvert A_i \rvert$	
(Higashi & Klir, 1982)	$m(\Theta) = 1$	$\log(\lvert\Theta\rvert)$
Yang and Han's method	$TU^I(m) = 1 - \frac{\sqrt{3}}{n} \sum_{\theta_i \in \Theta} d^I([Bel(\theta_i), Pl(\theta_i)], [0, 1])$	
(Yang & Han, 2016)	$m(\Theta) = 1$	1
Deng's measure	$TU_E^I(m) = \sum_{i=1}^{n} [1 - d_E^I([\mathrm{Bel}(\theta_i), Pl(\theta_i)], [0, 1])]$	
(Deng, 2018)	$m(\Theta) = 1$	$\lvert\Theta\rvert$
Deng entropy (Deng, 2016)	$E_d(m) = -\sum_{A_i \in 2^\Theta} m(A_i) \log \frac{m(A_i)}{2^{\lvert A_i\rvert} - 1}$	
(Kang & Deng, 2019)	$m(A) = \frac{2^{\lvert A_i\rvert-1}}{\sum_{A_i \in 2^\Theta} 2^{\lvert A_i\rvert} - 1}$	$\log(3^{\lvert\Theta\rvert} - 2^{\lvert\Theta\rvert})$
SU measurement	$SU(m) = \sum_{\theta_i \in \Theta} \left[-\frac{Pl(\theta_i)+Bel(\theta_i)}{2} \log \frac{Pl(\theta_i)+Bel(\theta_i)}{2} + Pl(\theta_i) - Bel(\theta_i) \right]$	
(Wang & Song, 2018)	$Bel(\theta_i) = 0, Pl(\theta_i) = 1$	$\lvert\Theta\rvert$
JS entropy	$JS(m) = \sum_{A_i \in 2^\Theta} m(A_i) \log(\lvert A_i \rvert) - \sum_{i=1}^{n} Pl_m(\theta_i) \log Pl_m(\theta_i)$	
(Jiroušek & Shenoy, 2018)	$m(\Theta) = 1$	$2\log(\lvert\Theta\rvert)$

The probability distribution of t_i based on Definition 6 of PTM is as follows.

$$Pl(\{t_1\}) = 1, \quad Pl(\{t_2\}) = 3/4, \tag{13}$$

$$Pl_m(\{t_1\}) = 4/7, \quad Pl_m(\{t_2\}) = 3/7. \tag{14}$$

Then we have

$$E_S(Pl_m(t_i)) = -4/7 \log(4/7) - 3/7 \log(3/7) \approx 0.985228. \tag{15}$$

Finally, the result of the proposed IVMF is

$$H_{IV'-PTM} = \sqrt{H_{IV} \cdot E_S(Pl_m(t_i))} \approx 1.825299. \tag{16}$$

Fig. 2 illustrates the splitting and calculation process of $H_{IV'-PTM}$. Dotted green arrows denote the splitting process of information volume of mass function and plausibility transformation method, while solid blue arrows indicate the calculation of H_{IV} and $E_S(Pl_m(t_i))$. The magnitude of H_{IV} and $E_S(Pl_m(t_i))$ are indicated in the figure using the lengths of the orange and cyan line segments, respectively. In the dashed circle with $H_{IV} + E_S(Pl_m(t_i))$ as the diameter, the magnitude of $H_{IV'-PTM}$ can be calculated by using the similarity of triangles in Eq. (17):

$$\frac{|H_{IV}|}{|H_{IV'-PTM}|} = \frac{|H_{IV'-PTM}|}{|E_S(Pl_m(t_i))|} \Rightarrow H_{IV'-PTM} = \sqrt{H_{IV} \cdot E_S(Pl_m(t_i))}. \tag{17}$$

Property 1. *The proposed method can be regarded as a geometric mean of first-order and higher-order of IVMF.*

Example 2. Given a FOD: $\Theta = \{t_1, t_2, \dots, t_n\}$, a BPA is defined as:

$m(\{t_i\}) = 1/n, \quad i = 1, 2, \dots, n.$

Then, the uncertainty in Example 2 is calculated as n increases from 1 to 11.

This example examines the suitability of the proposed approach when mass functions degenerate into a probability distribution. Table 2 delineates the uncertainty measures' outcomes for the BPAs. Furthermore, Fig. 3 visually depicts the trends of distinct uncertainty methods with changing n to provide a clear illustration.

Fig. 3 shows that the extent of uncertainty as quantified by Weighted Hartley entropy (Higashi & Klir, 1982) is always equal to 0, which means that this uncertainty measure is not valid in a probability distribution. One interesting point is that the result of Yang and Han's measure (Yang & Han, 2016), its result rises at first and then becomes smaller as n increases, which is counterintuitive. It can be explained from its formula: as the FOD increases, the mass function of each focal element decreases, resulting in a narrower belief interval assigned to each element and thus a smaller result.

Except for the two methods mentioned above and Deng's measure (Deng, 2018), all the uncertainty measures obtain the same result as Shannon entropy does. Therefore, the proposed method is acceptable when BPAs degenerate into a single focal element.

Property 2. *When the mass functions degenerate to a probability distribution, the proposed method degenerates to Shannon entropy.*

Example 3. Consider a FOD: $\Theta = \{t_1, t_2, \dots, t_n\}$, the mass function is given as $m(\{t_1, t_2, \dots, t_n\}) = 1$, with n grows from 1 to 11.

Fig. 4 shows the results of the uncertainty measures of the mass function. The results demonstrate that when a total uncertainty mass function is given, Hohle's (Höhle, 1982) and Yager's measures (Yager, 2008) are always equal to 0, while the methods proposed by Yang and Han (2016) and Higashi and Klir (1982) have a slight increase at the beginning, following by a decreasing growth rate as n increases, and finally slowly become a curve with a gradually decreasing slope.

As for the proposed method, the results are slightly higher than the Deng entropy (Deng, 2016) when n is very small. However, as n increases continuously, the uncertainty obtained by Deng entropy is higher than the former. This is also reasonable because when the n is big enough, a small increase of n would not make a huge difference. For example, when n is set to 1000 and 1001, a small change in the FOD does not result in a noteworthy difference in total uncertainty of mass function, especially in the case of vacuous BPA.

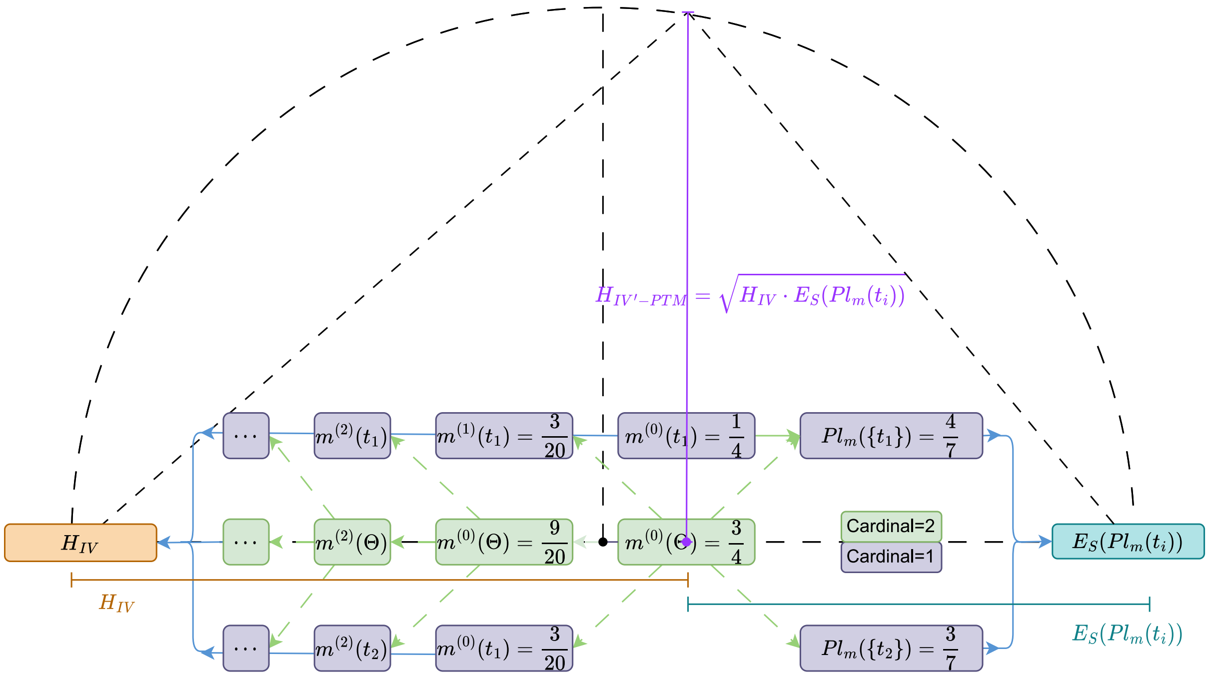


Fig. 2. The splitting and calculation process of $H_{IV'-PTM}$ in Example 1.

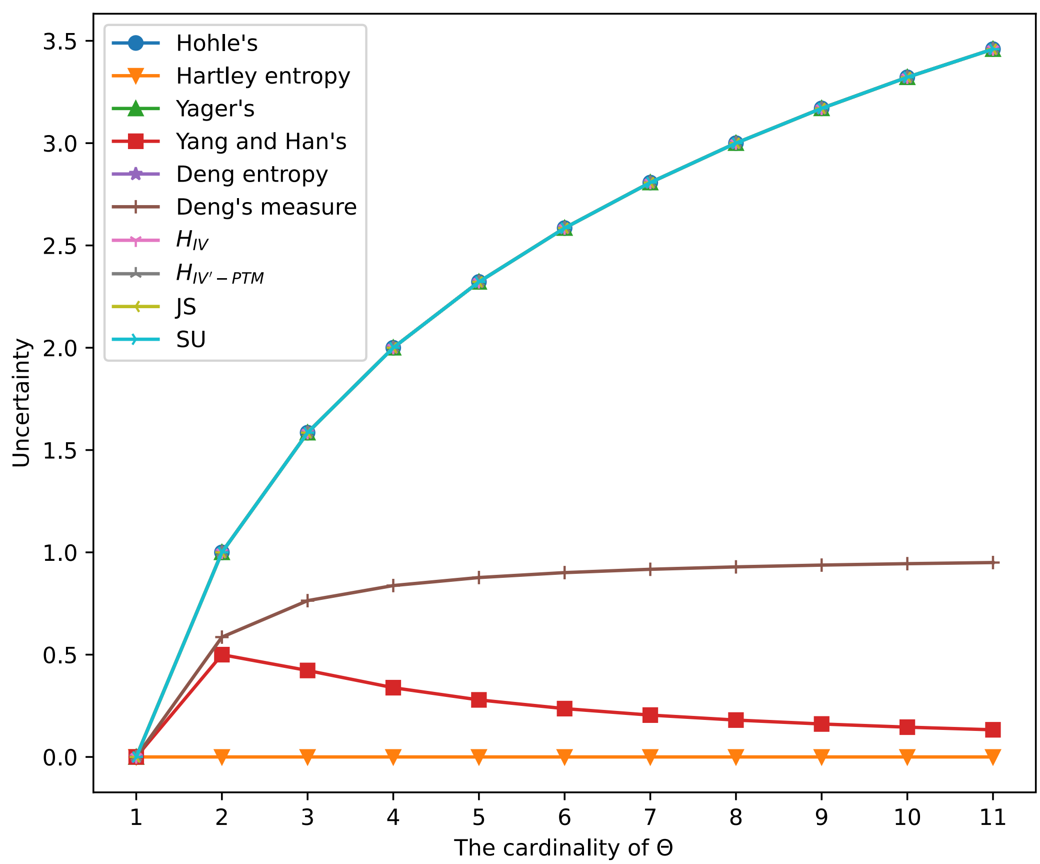


Fig. 3. The results of Example 2.

Another noticeable difference is that the IVMF proposed by Deng and Deng (2021) shows a much higher uncertainty as n increases. It means the proposed method entails a higher degree of consideration of additional information in the FOD. However, it can be seen from the

Table 2
The results of Example 2 when n changes from $n = 1$ to $n = 7$.

UM	Hohle's (Höhle, 1982)	Hartley entropy (Higashi & Klir, 1982)	Yager's (Yager, 2008)	Yang and Han's (Yang & Han, 2016)	JS entropy (Jiroušek & Shenoy, 2018)
n = 1	0.0000	0.0000	0.0000	0.0000	0.0000
n = 2	1.0000	0.0000	1.0000	0.5000	1.0000
n = 3	1.5850	0.0000	1.5850	0.4226	1.5850
n = 4	2.0000	0.0000	2.0000	0.3386	2.0000
n = 5	2.3219	0.0000	2.3219	0.2789	2.3219
n = 6	2.5850	0.0000	2.5850	0.2362	2.5850
n = 7	2.8074	0.0000	2.8074	0.2046	2.8074
UM	SU measurement (Wang & Song, 2018)	Deng's measure (Deng, 2018)	Deng entropy (Deng, 2016)	H_{IV} (Deng, 2020a)	$H_{IV'-PTM}$ (proposed method)
n = 1	0.0000	0.0000	0.0000	0.0000	0.0000
n = 2	1.0000	0.5858	1.0000	1.0000	1.0000
n = 3	1.5850	0.7639	1.5850	1.5850	1.5850
n = 4	2.0000	0.8377	2.0000	2.0000	2.0000
n = 5	2.3219	0.8769	2.3219	2.3219	2.3219
n = 6	2.5850	0.9010	2.5850	2.5850	2.5850
n = 7	2.8074	0.9172	2.8074	2.8074	2.8074

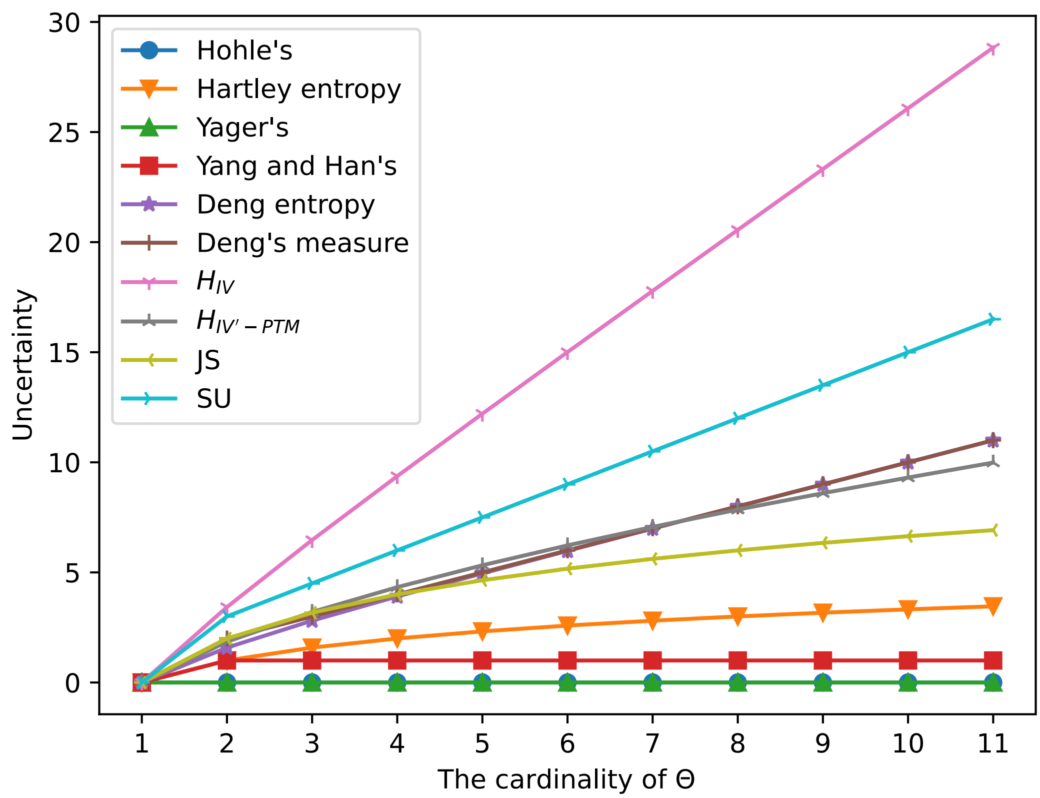


Fig. 4. The results of Example 3.

image that the slope of the curve of this method is almost constant as n increases, and in contrast, our proposed method is much more reasonable.

Example 4. Let $\Theta = \{t_1, t_2, t_3, t_4\}$ be the FOD. There exists two BPAs:

$m_1(\{t_1\}) = 1/4, \quad m_1(\{t_2\}) = 1/4, \quad m_1(\{t_1, t_2\}) = 1/2,$

$m_2(\{t_1\}) = 1/4, \quad m_2(\{t_2\}) = 1/4, \quad m_2(\{t_3, t_4\}) = 1/2.$

From the view of BPA, the FOD of these two mass functions m_1 and m_2 are $\Theta_1 = \{t_1, t_2\}$ and $\Theta = \{t_1, t_2, t_3, t_4\}$, respectively. This is the case when the FOD is inconsistent with each other. Uncertainty obtained by existing methods and the proposed method is shown in Fig. 5.

Intuitively, the uncertainties of m_1 and m_2 are different in this given FOD. Although the values of the two BPAs are the same, the uncertainty should be smaller for m_1 than for m_2. From the results in Fig. 6, only Yager's measure (Yager, 2008), Hohle's measure (Höhle, 1982), JS entropy (Jiroušek & Shenoy, 2018), SU measurement (Wang & Song, 2018), as well as $H_{IV'-PTM}$ reflected this result correctly, and the other entropy and methods did not get the correct result. Thus, when the BPAs values are the same, but the focal elements are different, the proposed method can correctly represent the differences and obtain intuitive results as JS entropy does.

The corresponding probability distribution of PPT and PTM in Example 4 are shown in Eqs. (18) and (19), respectively. Since probability transformation methods can be regarded as non-specificity loss (Zhou & Deng, 2022a), this method of equally assigning probability assignments appears to lead to a greater loss of non-specificity when a greater proposition is assigned to $\{t_3, t_4\}$, because intuitively t_3 and t_4 have a higher probability. Thus, PTM is used as our probability transformation

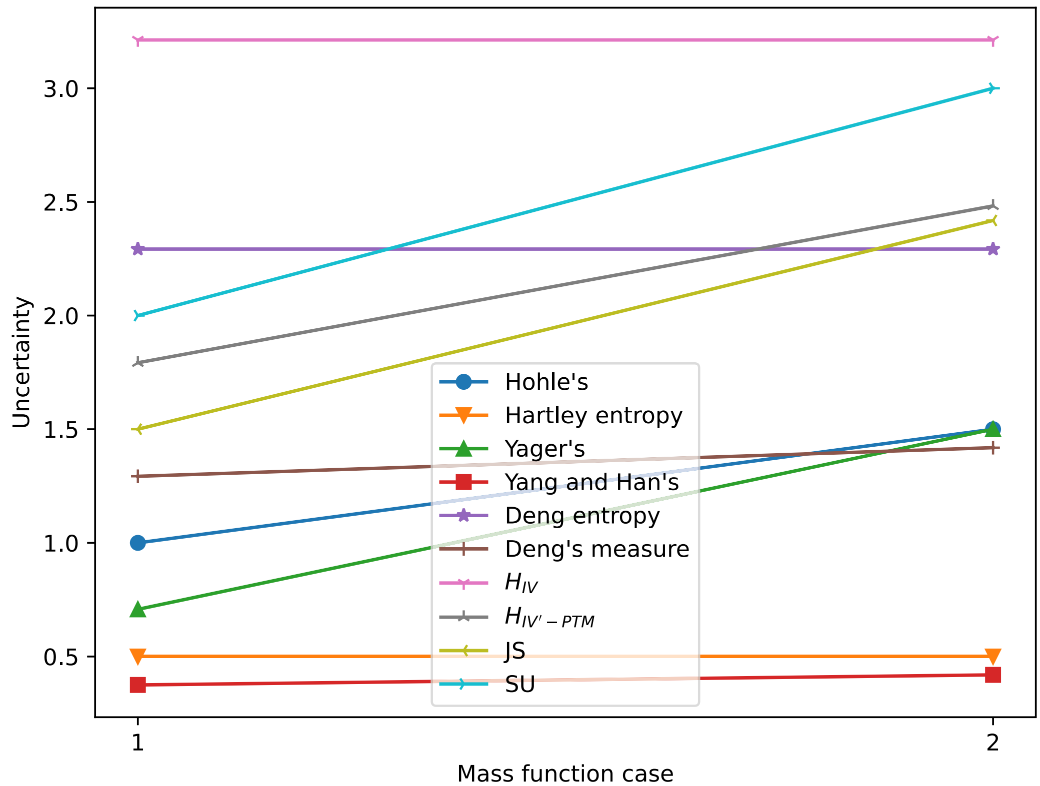


Fig. 5. The results of Example 4.

method in the proposed method instead of PPT.

$$m_2(\{t_1\}) = m_2(\{t_2\}) = m_2(\{t_3\}) = m_2(\{t_4\}) = 1/4, \tag{18}$$

$$m_2(\{t_1\}) = m_2(\{t_2\}) = 1/6, m_2(\{t_3\}) = m_2(\{t_4\}) = 1/3. \tag{19}$$

Example 5. Similar to Example 4, Let $\Theta = \{t_1, t_2, t_3, t_4\}$ be an FOD. Given two BPAs:

$m_1(\{t_1, t_2\}) = 0.3, \quad m_1(\{t_3, t_4\}) = 0.7,$

$m_2(\{t_1, t_2\}) = 0.3, \quad m_2(\{t_2, t_3\}) = 0.7.$

Example 5 is also used to test the discord and non-specificity measured by some existing entropy and methods as well as the proposed method when the FOD obtained from the BPAs is inconsistent. The cardinality of the mass function is set to the same to distinguish from Example 4. The results are shown in Fig. 6.

Among these 10 uncertainty measures listed in Fig. 6, only $H_{IV'-PTM}$, as well as JS entropy (Jiroušek & Shenoy, 2018) and SU measurement (Wang & Song, 2018), shows a lower uncertainty in the second case, which is coherent with intuitive. The remaining uncertainty measures cannot reflect this fact correctly and get the same result in these two BPAs.

As seen from Examples 4 and 5, when the FOD obtained from the mass functions is inconsistent, this inconsistency may be either with the given FOD or with the FOD obtained from different mass functions, leading to counterintuitive uncertainty. However, the proposed method can effectively and correctly obtain intuitive results.

Property 3. *Compared with the existing IVMF, the proposed method can get natural outcomes when the FOD conflict with each other.*

Example 6. This example is adapted from Deng (2020b) and used for testing the uncertainty as the discord and non-specificity changes. Let $\Theta = \{t_1, t_2, \dots, t_{15}\}$ be the FOD which contains 15 elements. A mass function is defined as

$m(\{t_4, t_5, t_6\}) = 0.15, \quad m(\{t_3\}) = 0.05, \quad m(T) = 0.7, \quad m(\{\Theta\}) = 0.1,$

Table 3
Results of Deng entropy, IVMF, and the proposed method in Example 6.

Cases	Deng entropy (Deng, 2016)	H_{IV} (Deng, 2020a)	$H_{IV'-PTM}$
T = $\{t_1\}$	3.2401	6.2753	4.6639
T = $\{t_1, t_2\}$	4.3496	8.6745	5.3973
T = $\{t_1, t_2, t_3\}$	5.2053	10.8043	5.9756
T = $\{t_1, \dots, t_4\}$	5.975	12.8353	6.4663
T = $\{t_1, \dots, t_5\}$	6.7081	14.8195	6.9393
T = $\{t_1, \dots, t_6\}$	7.4242	16.7791	7.4002
T = $\{t_1, \dots, t_7\}$	8.1322	18.7253	7.9121
T = $\{t_1, \dots, t_8\}$	8.8362	20.6638	8.4084
T = $\{t_1, \dots, t_9\}$	9.5382	22.5979	8.8906
T = $\{t_1, \dots, t_{10}\}$	10.2391	24.5296	9.3602
T = $\{t_1, \dots, t_{11}\}$	10.9396	26.4598	9.8182
T = $\{t_1, \dots, t_{12}\}$	11.6399	28.3893	10.2654
T = $\{t_1, \dots, t_{13}\}$	12.34	30.3182	10.7026
T = $\{t_1, \dots, t_{14}\}$	13.0401	32.247	11.1305

where T is a variable subset of Θ with only one element with its cardinality increasing from 1 to 14. Namely, the number of the elements of T is changing from 1 to 14 by adding element $t_1, t_2, \dots, t_{14}$ to T.

When T changes, uncertainty measured by given uncertainty measures is shown in Fig. 7. Table 3 shows the results between Deng entropy (Deng, 2016), IVMF (Deng, 2020a), and the proposed method.

As seen from Fig. 7, Yager's (Yager, 2008) and Hohle's measures (Höhle, 1982) decline as the cardinality of T rises, which contradicts the intuition that as T increases, uncertainty also increases.

It is reasonable that the uncertainty of BPA increases as the cardinality of T grows, for the fact that T gradually intersects with other propositions, which adds the uncertainty intuitively.

Table 3 shows different results among the three measures. The two kinds of information volume give a higher value of Deng entropy (Deng, 2020a), this is reasonable since the two measures consider a higher order of information. However, as the cardinality of T increases, the

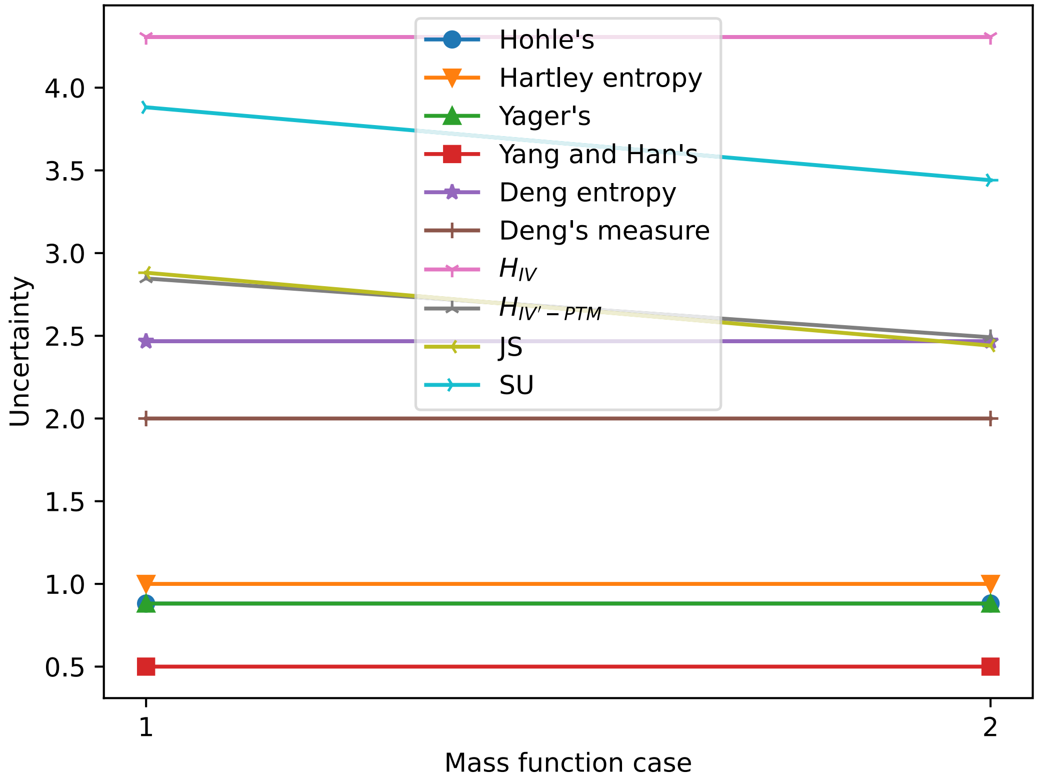


Fig. 6. The results of Example 5.

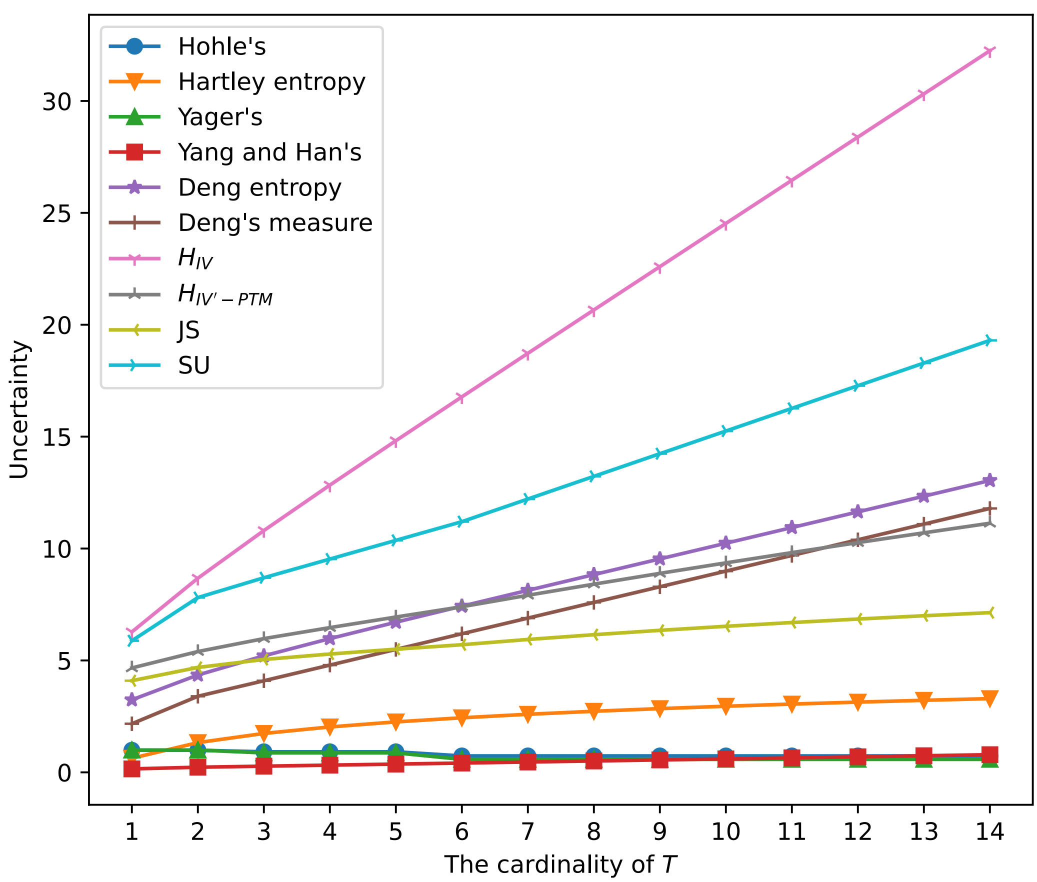


Fig. 7. The results of Example 6.

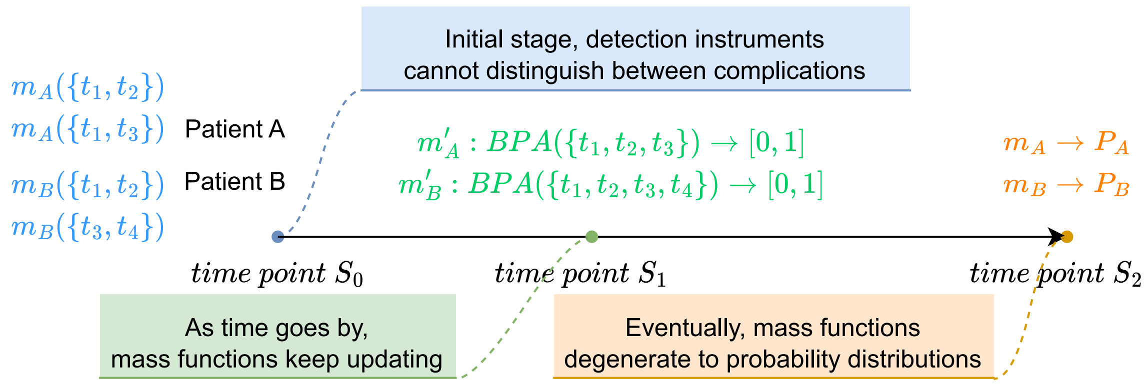


Fig. 8. The development of complications detection on patient A, B.

results given by the proposed measure are less than Deng entropy, this can also be explained in Example 3. When the cardinality of T is small, its elements have no intersection with other propositions, resulting in a higher uncertainty than Deng entropy.

5. Application in threat assessment

In this section, the proposed method will be applied to disease threat assessment to illustrate the performance of the proposed method.

5.1. Problem statement

Human immunodeficiency virus (HIV) causes defects in the human immune system and is also known as the Acquired Immunodeficiency Syndrome (AIDS) virus. AIDS is a major global public health issue. HIV infection destroys the body's immune system, leading to various complications and ultimately the death of the infected individuals. The World Health Organization (WHO) classifies HIV into four stages, with most immunodeficiency-related complications occurring in the last two stages.

Suppose two HIV patients, A and B, arrive at the hospital, both in the third stage of HIV. Initially, there are no evident complication symptoms in the patients, and complications may have similar symptoms. The diagnostic instruments can only provide an estimated probability of a patient experiencing a particular symptom. For convenience, t_1, t_2, t_3, t_4 are used to represent the four major common complications, and it is assumed that these four symptoms have the same severity. The higher the number of complications that a patient experiences, the greater the threat level of HIV in their body. The threat degree of a specific symptom is considered as the probability of diagnosing that symptom, which can be represented by a mass function $m : BPA(\{t_1,t_2,t_3,t_4\}) \to [0,1]$. For example, assume that the initial complications probabilities for patients A and B are represented by m_A and m_B, respectively.

$$m_A : m_A(\{t_1,t_2\}) = 0.7, \quad m_A(\{t_1,t_3\}) = 0.3, \tag{20}$$

$$m_B : m_B(\{t_1,t_2\}) = 0.7, \quad m_A(\{t_3,t_4\}) = 0.3. \tag{21}$$

m_A and m_B indicate that patient A has a 70% chance of being diagnosed with complication t_1 or t_2, and a chance of 30% to be diagnosed with complication t_1 or t_3. And patient B has a chance of 70% to be diagnosed with complication t_1 or t_2 due to some suspicious symptoms, and 30% of being diagnosed with complication t_3 or t_4 due to other symptoms.

After a period of treatment and observation, the symptoms keep developing and are clearer, and finally, these mass functions will degenerate to probability distributions. The entire developing process is shown in Fig. 8.

Table 4
The magnitude of mass functions m_A and m_B in different situations, where $m(\{t_1,t_2\}) = m_A(\{t_1,t_2\}) = m_B(\{t_1,t_2\})$.

$m(\{t_1,t_2\})$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$m_A(\{t_1,t_3\})$	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1
$m_B(\{t_3,t_4\})$	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	0.1

5.2. A possible physical explanation of IVMF

The threat ranking should be proportional not only to the number of complications, but also to the uncertainty in diagnosing the complications. Obviously, the information volume of mass function is suitable to quantify the severity here. A higher information volume indicates a more serious threat.

Now the question is how to measure the information volume of a certain period at time point S_0 ? For a certain time point on the timeline, uncertainty can be measured by uncertainty measures. But when it comes to covering a period of time, most of them may be unsuitable. Since the fractal idea of continuously splitting mass function into its power set simulates the evolution of BPA (Zhou & Deng, 2022b), the splitting process of IVMF can be regarded as splitting time into numbers of segments to measure the information volume of a certain period. Thus, splitting-based uncertainty measures like IVMF are efficacious here.

Intuitively, the symptom is clearer as time goes by, which means the proportion of compound elements decreases, while the proportion of single focal elements increases, and this can be expressed by the splitting process. Fig. 9 illustrates this idea.

Based on the above, IVMF and the proposed method can be applied to disease threat assessment.

5.3. Experiment and results

To better explore how changes in the probability of symptom diagnosis affect the threat assessment of the disease, the mass functions m_A and m_B in the initial stage are set to different values and analyzed. The values and results are shown in Table 4 and Fig. 10.

As shown in Fig. 10, as the magnitude of $m(\{t_1,t_2\})$ increases, both IVMF and the proposed method will first increase, reaching a maximum value at $m(\{t_1,t_2\}) = 0.5$, and then will decrease. This can be explained by the fact that when the magnitude of each BPA is the same, the patients A, B have a maximum chance of being diagnosed with a maximum number of complications. Besides, all four broken lines in the line chart are symmetric about $m(\{t_1,t_2\}) = 0.5$, this is intuitive since the cardinality of elements in each BPA is the same. Unlike Deng (2020a)' s IVMF whose results are consistent between patient A and patient B, the proposed method can perform a better threat assessment. Because intuitively the more complications the patient is diagnosed

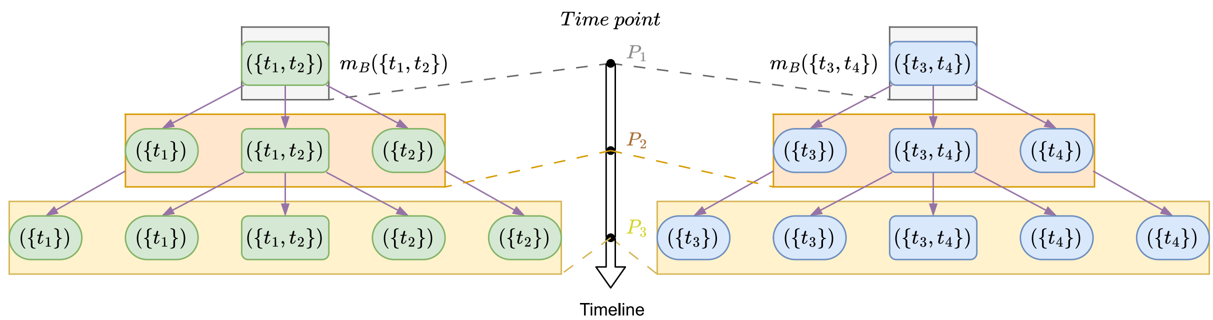


Fig. 9. The splitting process of mass function on patient B.

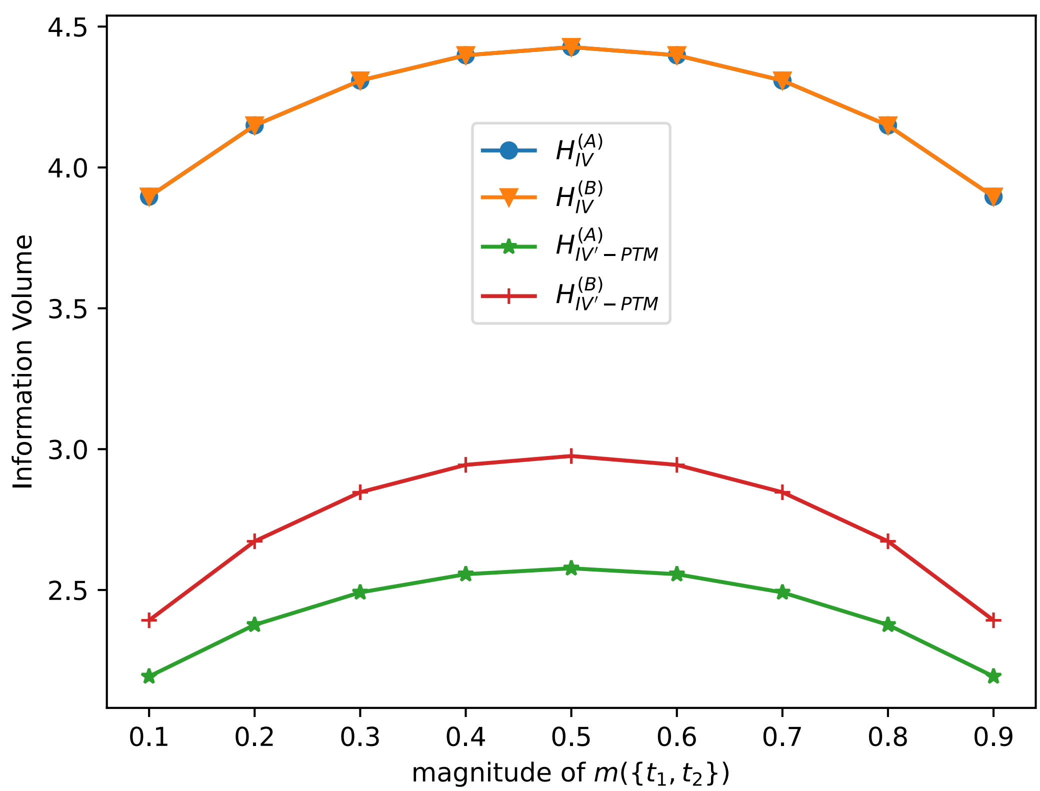


Fig. 10. Disease threat assessment obtained by IVMF (H_{IV}) and the proposed method ($H_{IV'-PTM}$). Superscript (A) and (B) refer to patient A and B, respectively.

with, the higher risk the person is. Moreover, the difference between $H_{IV'-PTM}^{(A)}$ and $H_{IV'-PTM}^{(B)}$ reaches its maximum at $m(\{t_1, t_2\}) = 0.5$. This is consistent with intuition because patient B always has a greater probability of being diagnosed with more complications, which reaches its maximum at $m(\{t_1, t_2\}) = 0.5$.

Apart from that, other uncertainty measures mentioned previously are also compared with the proposed method. Noting that these measures are different from IVMF, which measures uncertainty using the splitting method to simulate the evolution of BPA as time goes by, while they may not reflect this feature correctly, some of them do obtain intuitive results.

As shown in Fig. 11, when the magnitude of $m_A(\{t_1, t_2\})$ increases, Hartley entropy (Higashi & Klir, 1982), Yang and Han's measure (Yang & Han, 2016), and Deng's measure (Deng, 2018) hardly or even not reflect this change, while other methods listed in the legend catch it exactly. Moreover, when applying these methods to both patients A and B, the IVMF proposed by Deng (2020a), as well as Deng entropy (Deng, 2016), cannot distinguish between A and B, which is counterintuitive. Although other methods such as Hohle's measure (Höhle, 1982), Yager's measure (Yager, 2008), JS (Jiroušek & Shenoy, 2018) and SU (Wang & Song, 2018) mirror this difference due to incomplete FOD, their calculation process do not contain explainable physical meaning as the proposed method does (see Fig. 12).

In general, this section can be concluded that:

- Uncertainty measures based on the splitting method can be used for measuring the information volume of a certain period.
- Compared with the existing IVMF, the proposed method can perform a better threat assessment when the FODs in different mass functions are inconsistent.
- The calculation process of the proposed method reflects the evaluation of BPAs as time goes by.

6. Conclusion

Information volume, or Shannon entropy, is an established metric in probability theory used to determine the amount of information contained within a probabilistic event or probability distribution. To quantify the uncertainty of a mass function within a power set, Deng (2020a) proposed the information volume of mass function (IVMF) based on Deng entropy (Deng, 2016). While sharing some properties with Shannon entropy for certain cases, the existing IVMF proves

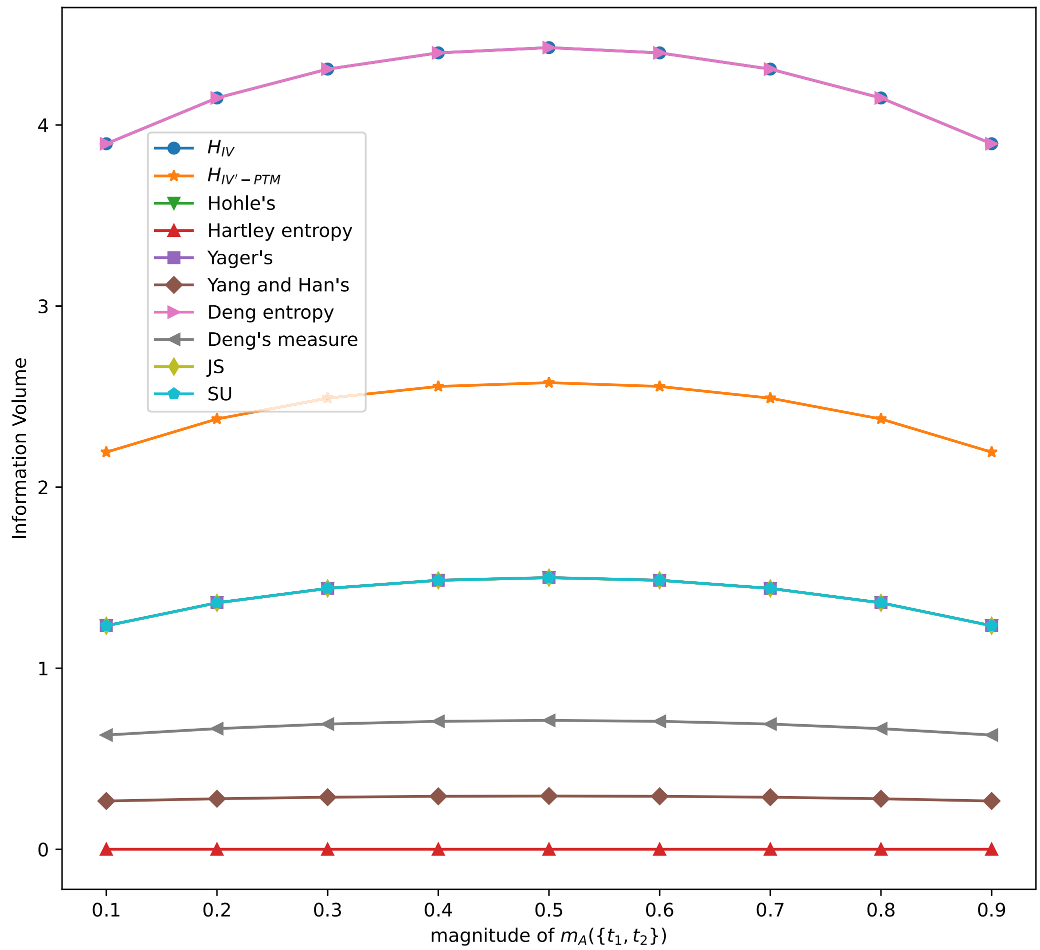
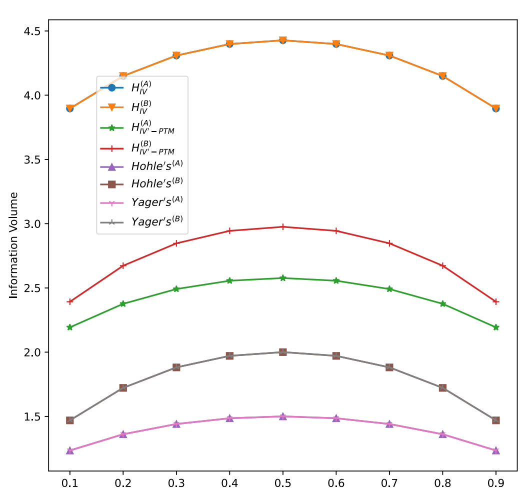
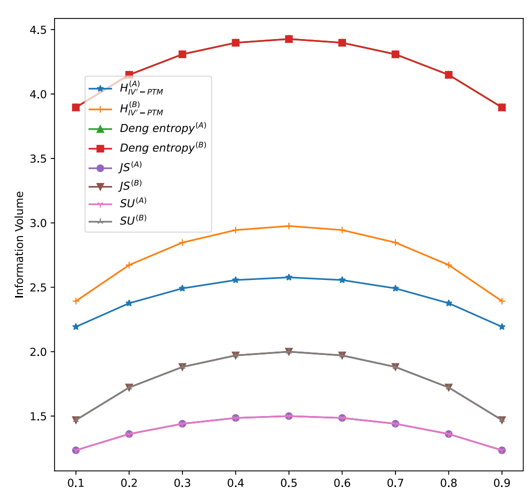


Fig. 11. Results obtained by the proposed method ($H_{IV'-PTM}$) and other uncertainty methods in disease threat assessment of patient A.



(a) Results (part a).



(b) Results (part b).

Fig. 12. Results obtained by the proposed and other methods in disease threat assessment of patients A and B.

unreasonable under some circumstances, e.g. for two basic probability assignments (BPAs) with a distinct frame of discernment (FOD). To address this issue, an improved IVMF is presented in this paper. The proposed measure can be regarded as a geometric mean of first-order information volume and higher-order information volume. When BPA degenerates to a probability distribution, the proposed method can degenerate to Shannon entropy. This paper presents several numerical examples to demonstrate that the proposed approach effectively addresses the issue of inconsistency of the FOD in BPA. Furthermore, a comparative analysis is conducted on several other extant measures of uncertainty. The result in these numerical examples shows the rationality and efficacy of the proposed method as a veritable measure of uncertainty. And eventually, the proposed method is validated in the real-world application of threat assessment.

CRediT authorship contribution statement

Jiefeng Zhou: Conceptualization, Methodology, Formal analysis, Investigation, Writing – original draft, Writing – review & editing. **Zhen Li:** Discussion, Validation. **Yong Deng:** Writing – review & editing, Supervision, Project administration, Funding acquisition.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request

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