Limit of the Maximum Random Permutation Set Entropy

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Abstract

The Random Permutation Set (RPS) is a recently proposed new type of set, which can be regarded as the generalization of evidence theory. To measure the uncertainty of RPS, the entropy of RPS and its corresponding maximum entropy have been proposed. Exploring the maximum entropy provides a possible way to understand the physical meaning of RPS. In this paper, a new concept, the *envelope* of entropy function, is defined. In addition, the limit of the *envelope* of RPS entropy is derived and proved. Compared with the existing method, the computational complexity of the proposed method to calculate the *envelope* of RPS entropy decreases greatly. The result shows that when the cardinality of a RPS (marked as N) approaches to infinity, the limit form of the *envelope* of the entropy of RPS converges to $\boldsymbol{e} \cdot (\mathrm{N}!)^{\mathbf{2}}$, which is highly connected to the constant $\boldsymbol{e}$ and factorial. Finally, numerical examples validate the efficiency and conciseness of the proposed *envelope*, which provides a new insight into the maximum entropy function.

Keywords: Shannon entropy, Deng entropy, Dempster–Shafer evidence theory, Approximation, Random permutation set, Maximum entropy

1 introduction

Uncertainty management is a significant issue that has attracted a lot of interest in various kinds of research fields. The classical tool to deal with uncertainty is probability theory (Jaynes, 2003), which allocates the probability distribution defined in a mutually exclusive event space. However, a particular challenge arises when uncertain information needs to be defined on the power set of the event space, which cannot be effectively handled by probability theory. To address this issue, Dempster-Shafer evidence theory (DSET) (Dempster, 2008; Shafer, 1976) is developed. DSET generalized probability theory by extending the probability distribution to a mass function defined on the collection of all possible subsets. As a result, this extension enabled the application of DSET to domains characterized by a high degree of complexity and uncertainty (Xiao, 2023; Chen and Deng, 2024). But both theories overlook the significance of considering ordered information in processing uncertain information This aspect, however, holds substantial importance and should not be ignored. Thus, Deng (2022) proposed the random permutation set (RPS), considering the ordered information while fully compatible with DSET and probability theory.

To measure the uncertainty, Shannon entropy (Shannon, 1948) is used in probability theory. In DSET, Deng (2016) proposed Deng entropy. In RPS, Chen and Deng (2023) proposed RPS entropy. Each entropy offers an efficient approach to understanding uncertainty. The maximum entropy principle was extensively studied by Jaynes (1957, 1982). The principle asserts that the distribution with the maximum entropy is the most appropriate representation of a system's current state. For convenience, this paper defines the concept of *envelope* for entropy. It refers to the function inside

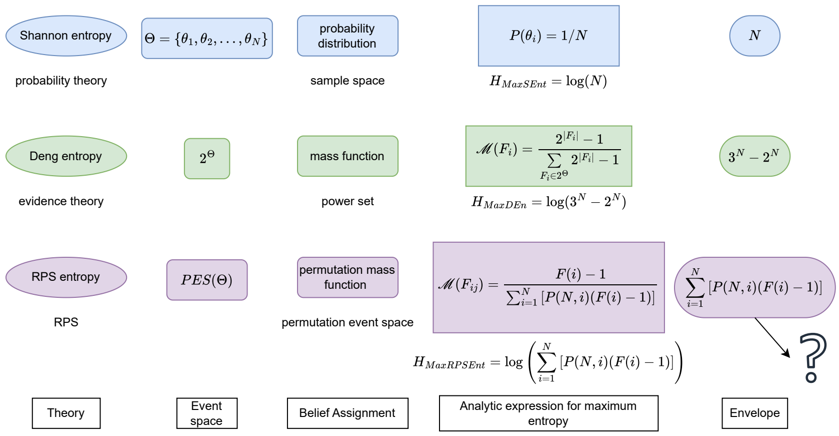

Fig. 1 The connection between Shannon entropy (Shannon, 1948), Deng entropy (Deng, 2016), and RPS entropy (Chen and Deng, 2023). The *envelope* is the function in the logarithmic function in maximum entropy expression.

the logarithmic function in the overall entropy expression for the maximum entropy case of a system. This concept becomes particularly significant when considering specific examples or scenarios. For instance, in a thermodynamic system, the *envelope* of entropy can represent the range of possible energy states that lead to the highest level of disorder or randomness. Understanding the *envelope* of entropy provides valuable insights into the behavior and characteristics of the system under consideration. As shown in Fig. 1, for a sample space whose cardinality is N, the *envelope* of Shannon entropy of a probability distribution is N, while in a power set whose maximum cardinality of its subsets is N, the *envelope* of Deng entropy is $3^N - 2^N$. Although the analytical expression of the *envelope* of RPS entropy is given by Deng and Deng (2022), its computational and expression complexity makes it difficult and inconvenient to discuss the properties and its physical meaning of maximum RPS entropy.

To address this issue, the limit form of the *envelope* of RPS entropy is presented and proved in this paper. The limit $e \cdot (N!)^2$ is very concise and related to the natural constant e and factorial, whose physical meaning may be an interesting topic in future research. Besides, numerical analysis validates the conciseness and correctness of the result.

The remainder of this paper is structured as follows. First, Sec. 2 introduces preliminary information and definitions related to the work. Next, Sec. 3 presents the definition and an illustrative example of the concept of *envelope*. After that, Sec. 4 provides and proves the limit form of the entropy *envelope* for the RPS entropy. To supplement this theoretical analysis, Sec. 5 then gives a comparative analysis between three types of maximum entropies. Finally, Sec. 6 summarizes the key conclusions of the paper.

2 Preliminaries

Some preliminaries are introduced in this section.

2.1 Mass function in power set

To better model and reason uncertainty in the real world, many theories and approaches have been proposed to address this issue, among which Dempster-Shafer evidence theory (DSET) (Dempster, 2008; Shafer, 1976) is an effective method for uncertainty reasoning and information fusion, which also allows one to generate information measures for uncertainty quantification (Contreras-Reyes and Kharazmi, 2023; Kharazmi and Contreras-Reyes, 2024, 2023).

2.1.1 Power set

Let Ω be the frame of discernment (FOD), expressed as $\Omega = \{\chi_1, \chi_2, \ldots, \chi_N\}$, whose elements are mutually exclusive and exhaustive. Its corresponding **power set** 2^Ω contains all the subsets of Ω, denoted as

$$2^\Omega = \{\emptyset, \{\chi_1\}, \{\chi_2\}, \ldots, \{\chi_N\}, \{\chi_1, \chi_2\}, \{\chi_1, \chi_3\}, \ldots, \Omega\}. \tag{1}$$

The cardinality of 2^{Ω} is 2^N where N is the cardinality of Ω, and this is highly related to the Sierpinski gasket, a recent study proposed by Zhou and Deng (2023)also reveals this connection.

2.1.2 Mass function

The **mass function** or basic probability assignment (BPA), is a mapping $\mathscr{M} : 2^{\Omega} \to [0, 1]$ (Dempster, 2008; Shafer, 1976) with bound conditions:

$$\mathscr{M}(\emptyset) = 0, \quad \sum_{M_i \in 2^{\Omega}} \mathscr{M}(M_i) = 1. \tag{2}$$

2.2 Deng entropy

To measure the uncertainty of a given system, numerous methodologies and entropy measures have been proposed. Among these, Deng entropy (Deng, 2016) has faced criticism (Abellán, 2017; Moral-García and Abellán, 2020); however, it continues to flourish in various domains and theoretical frameworks, including prior distributions (Li and Xiao, 2023), linearity phenomena Zhao et al. (2024a), interactive systems (Wang et al., 2024), and game theory (Chen et al., 2024). Furthermore, it has inspired the development of alternative information measures (Contreras-Reyes and Kharazmi, 2023; Kharazmi and Contreras-Reyes, 2024, 2023).

2.2.1 Deng entropy

Given a mass function $\mathscr{M}(2^{\Omega})$, **Deng entropy** (Deng, 2016) is defined as

$$H_{DEnt}(\mathscr{M}) = -\sum_{M_i \in 2^{\Omega}} \mathscr{M}(M_i) \log \frac{\mathscr{M}(M_i)}{2^{|M_i|} - 1}, \tag{3}$$

where $|M_i|$ is the cardinality of M_i. When the element in each mass function is a singleton set, i.e. $\forall M_i \in 2^{\Omega}, |M_i| = 1$, then Deng entropy degenerates into **Shannon**

entropy (Shannon, 1948):

$$H_{SEnt}(P) = -\sum_{p_i \in P} p_i \log(p_i). \tag{4}$$

Theorem 1 (Maximum Deng entropy (Kang and Deng, 2019))**.** *Given a FOD* Ω*, the* ***maximum Deng entropy*** $H_{MaxDEnt}$ *can be obtained when its mass function has the following form:*

$$\mathscr{M}(M_i) = \frac{2^{|M_i|} - 1}{\sum\limits_{M_i \in 2^\Omega} 2^{|M_i|} - 1}. \tag{5}$$

Its corresponding Deng entropy reaches its maximum:

$$\begin{aligned} H_{MaxDEnt} &= \sum_{M_i \in 2^\Omega} \mathscr{M}(M_i) \log \frac{\mathscr{M}(M_i)}{2^{|M_i|} - 1} \\ &= \log \sum_{M_i \in 2^\Omega} (2^{|M_i|} - 1). \end{aligned} \tag{6}$$

2.3 Random Permutation Set

Random Permutation Set (RPS) has been presented as a means to manage uncertainty with ordered information (Deng, 2022). RPS entails permuting items in a given set, allowing for effective handling of uncertainty in ordered data. Based on the above, some works like reasoning under the framework of RPS (Deng et al., 2024. DOI: 10.1109/TPAMI.2024.3438349), fusion order (Zhou et al., 2024; Wang et al., 2024), generalized information entropy (Zhan et al., 2024), and random walk model (Zhou et al., 2024b) are developed. To better illustrate our work, some fundamental definitions of RPS are introduced briefly here.

2.3.1 Permutation Event Space (PES)

Given a finite set with N elements $\Omega = \{\chi_1, \chi_2, \ldots, \chi_N\}$, its **Permutation Event Space (PES)** (Deng, 2022) is a **ordered** set containing all possible permutations of all subsets of Ω, and is given by

$$\begin{aligned}
PES(\Omega) = &\left\{M_{i,j} \mid i = 0, \ldots, N; j = 1, \ldots, A_N^i\right\} \\
= &\{\emptyset, (\chi_1), (\chi_2), \ldots, (\chi_N), (\chi_1, \chi_2), \\
&(\chi_2, \chi_1), \ldots, (\chi_{N-1}, \chi_N), (\chi_N, \chi_{N-1}), \\
&\ldots, (\chi_1, \chi_2, \ldots, \chi_N), \ldots, (\chi_N, \chi_{N-1}, \ldots, \chi_1)\},
\end{aligned} \tag{7}$$

where $A_N^i = N!/(N-i)!$ is the number of choices to select i ordered elements from a collection with N elements. The element $M_{i,j}$ is called **permutation event**.

2.3.2 Random Permutation Set (RPS)

Given a finite set with N elements $\Omega = \{\chi_1, \chi_2, \ldots, \chi_N\}$, its **Random Permutation Set (RPS)** (Deng, 2022) is a set of pairs given by

$$RPS(\Omega) = \{\langle M_{i,j}, \mathcal{M}(M_{i,j})\rangle \mid M_{i,j} \in PES(\Omega)\}, \tag{8}$$

where $\mathcal{M}$ is called **permutation mass function (PMF)**, a mapping $PES(\Omega) \to [0, 1]$ with bound conditions

$$\mathcal{M}(\emptyset) = 0, \qquad \sum_{M_{i,j} \in PES(\Omega)} \mathcal{M}(M_{i,j}) = 1. \tag{9}$$

RPS is completely consistent with DSET and probability theory. When the order of elements within PES is disregarded, PES effectively becomes a power set, while the PMF within RPS reduces to the mass function. Moreover, if each permutation event contains only a singular element, RPS simplifies into probability theory, wherein PES degenerates into the sample space. Fig. 2 illustrates the connection between RRS, DSET, and probability theory.

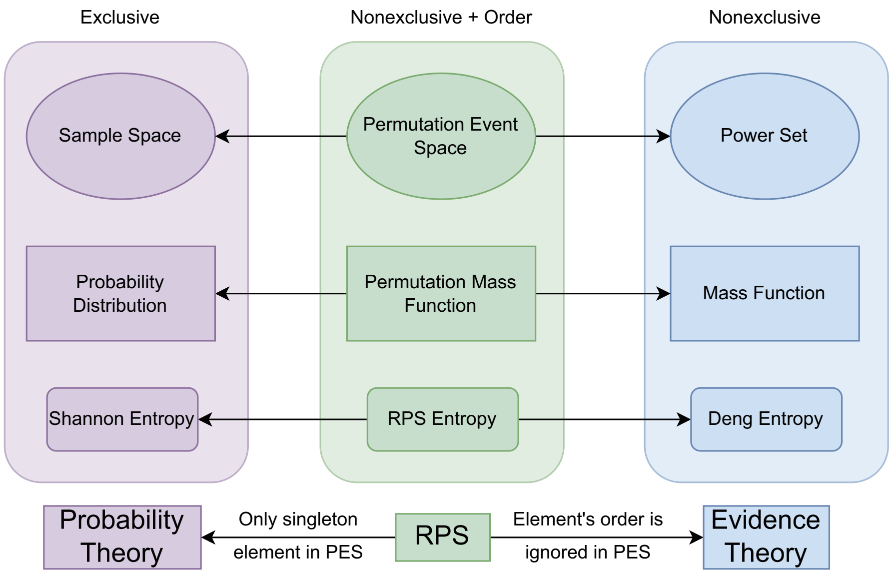

Fig. 2 The relationship between RPS, DSET and probability theory.

To ascertain the degree of uncertainty in RPS, the entropy of RPS, as introduced by Chen and Deng (2023), plays a significant role, exhibiting compatibility with both Deng entropy and Shannon entropy.

2.3.3 RPS entropy

For a RPS $RPS(\Omega) = \{\langle M_{i,j}, \mathcal{M}(M_{i,j})\rangle \mid M_{i,j} \in PES(\Omega)\}$, defined on a PES: $PES(\Omega) = \left\{M_{i,j} \mid i = 0, \ldots, N; j = 1, \ldots, A_N^i\right\}$, the entropy of RPS (Chen and Deng, 2023) is defined as

$$H_{RPSEnt}(\mathcal{M}) = -\sum_{i=1}^{N}\sum_{j=1}^{A_N^i} \mathcal{M}\left(M_{i,j}\right)\log\left(\frac{\mathcal{M}\left(M_{i,j}\right)}{S_A(i)-1}\right), \tag{10}$$

where $S_A(i) = \sum_{a=0}^{i} A_i^a = \sum_{a=0}^{i} \frac{i!}{(i-a)!}$ is the sum of all possible permutations of a finite set containing i elements.

Though the entropy of RPS is introduced, there is a lack of in-depth analysis regarding its maximum entropy. To address this issue, Deng and Deng (2022) presented an analytical expression for the maximum entropy of RPS as well as its corresponding PMF condition.

Understanding the maximum RPS entropy is crucial as it represents the state of maximum uncertainty within the framework, providing an important theoretical boundary for uncertainty quantification. This maximum value serves as a reference point for comparing different RPS configurations, and helps establish the framework's capacity for uncertainty representation. The following theorem presents this maximum value and its corresponding conditions.

Theorem 2 (Maximum RPS entropy (Deng and Deng, 2022))**.** *Given a PES:* $PES(\Omega) = \left\{M_{i,j} \mid i = 0, \ldots, N; j = 1, \ldots, A_N^i\right\}$*, if and only if its PMF satisfies*

$$\mathcal{M}(M_{i,j}) = \frac{S_A(i)-1}{\sum_{i=1}^{N}\left[A_N^i(S_A(i)-1)\right]}. \tag{11}$$

Then the entropy of RPS reaches its maximum:

$$H_{MaxRPS} = \log\left(\sum_{i=1}^{N}\left[A_N^i(S_A(i)-1)\right]\right). \tag{12}$$

3 The envelope of entropy

In this section, the definition of the *envelope* of entropy is given, followed by an example as an illustration.

Within a system, the value of entropy can vary across multiple situations. In this context, the *envelope* of entropy refers to the function encapsulated within the logarithmic expression of entropy that reaches its maximum value.

3.1 The definition of envelope of entropy

For a given definition of entropy $H_{Ent} = \mathbb{E}(-\log(f(P)))$, where P is a belief assignment within a system $\mathcal{S}$ and $\mathbb{E}(X)$ is the mathematical expectation of X. Then the *envelope* of H_{Ent} can be defined as

$$H_{MaxEnt} = \max_{P \in \mathcal{S}} \left[\mathbb{E}(-\log(f(P)))\right], \tag{13}$$

$$C_e(H_{Ent}) = \exp(H_{MaxEnt}), \tag{14}$$

where H_{MaxEnt} is the maximum entropy for a given system, and C_e is the *envelope* of a given entropy H_{Ent}.

To illustrate the practical application of this theoretical framework, we examine two fundamental entropy measures. This example aims at demonstrating how the envelope concept manifests in both classical entropy in information theory and its modern extensions to uncertainty quantification, providing concrete insights into the behavior of entropy across different mathematical structures.

Example 1 (The envelope of Shannon entropy and Deng entropy)**.** *Given a finite set* $\Omega = \{\chi_1, \chi_2, \ldots, \chi_N\}$*, we compute*

1. *the envelope of Shannon entropy (Shannon, 1948);*
2. *the envelope of Deng entropy (Deng, 2016) in power set* 2^{Ω}*.*

For a finite set Ω, Shannon entropy reaches its maximum when the probability distribution is uniform, i.e. $P(\chi_i) = 1/N$. Then Eq.(4) can be rewritten as

$$H_{SEnt}(P) = \log(N). \tag{15}$$

Based on Eq.(14), the *envelope* of Shannon entropy is

$$C_e(H_S) = \exp\left(\log(N)\right) = N. \tag{16}$$

If the mass function satisfies (Kang and Deng, 2019)

$$\mathscr{M}(M_i) = \frac{2^{|M_i|} - 1}{\sum_{M_i \in 2^\Omega} 2^{|M_i|} - 1}, \tag{17}$$

then Deng entropy reaches its maximum, which can be simplified as (Qiang et al., 2023)

$$\begin{aligned}
H_{MaxDEnt} &= \sum_{M_i \in 2^\Omega} \mathscr{M}(M_i) \log \frac{\mathscr{M}(M_i)}{2^{|M_i|} - 1} \\
&= \log \sum_{M_i \in 2^\Omega} (2^{|M_i|} - 1) \\
&= \log \left(\sum_{a=0}^{N} (C_N^a (2^a - 1)) \right) \\
&= \log \left(3^N - 2^N\right),
\end{aligned} \tag{18}$$

where $C_N^a = N!/[(N-a)! \cdot a!]$ is the combination number.

Based on Eq.(14), the *envelope* of Deng entropy is

$$C_e(H_{DEnt}) = \exp\left(\log\left(3^N - 2^N\right)\right) = 3^N - 2^N. \tag{19}$$

As shown in Fig. 3, given a uniform probability distribution, H_{SEnt} reaches its maximum $\log(N)$, then the *envelope* of H_{SEnt} is N. When the mass function satisfies $\mathscr{M}(M_i) = 2^{|M_i|} - 1/[\sum_{M_i \in 2^\Omega} 2^{|M_i|} - 1]$, H_{DEnt} reaches its maximum: $\log\left(3^N - 2^N\right)$, then the *envelope* of H_{DEnt} is $\left(3^N - 2^N\right)$.

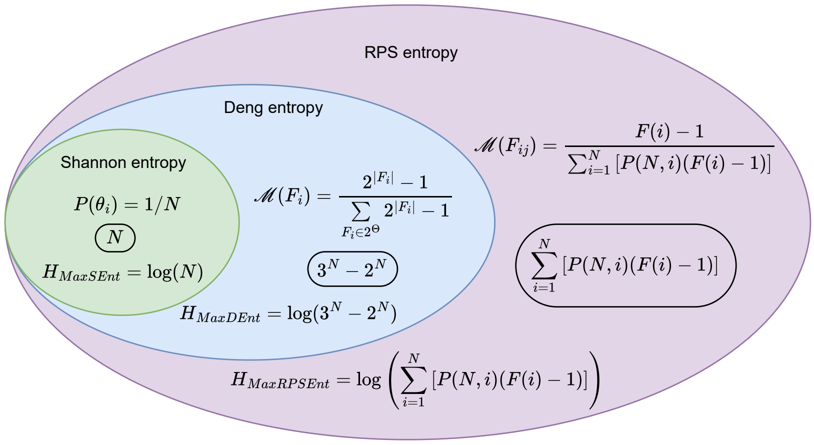

Fig. 3 The *envelope* of Shannon entropy, Deng entropy and RPS entropy.

The preceding analysis demonstrates how the envelope concept manifests among Shannon entropy, Deng entropy and RPS entropy, revealing their distinct characteristics in measuring uncertainty. Then we can establish a comprehensive framework for comparing these different uncertainty measures. The following theorem formalizes these results by presenting the analytical expressions for all three entropy envelopes in a unified form.

Theorem 3 (Envelope of entropy)**.** *The envelope of Shannon entropy, Deng entropy, and RPS entropy are presented as*

$$\textit{Shannon entropy}: C_e(H_{SEnt}) = N, \tag{20}$$

$$\textit{Deng entropy}: C_e(H_{DEnt}) = 3^N - 2^N, \tag{21}$$

$$\textit{RPS entropy}: C_e(H_{RPSEnt}) = \sum_{a=1}^{N} \left[A_N^a (S_A(a) - 1)\right]. \tag{22}$$

4 Limit of the envelope of RPS entropy

The proof of the limit of the *envelope* of RPS entropy is presented in this section. Let A_N^u marked as the u-permutation of N, and $S_A(N)$ marked as the sum of A_N^u when u change from 0 to N. Thus

$$A_N^u = \frac{N!}{(N-u)!}, \tag{23}$$

$$S_A(N) = \sum_{u=0}^{N} A_N^u = \sum_{u=0}^{N} \frac{N!}{(N-u)!}. \tag{24}$$

Lemma 1 (sum of permutation)**.** *When $N \geqslant 1$, $S_u(N)$ can be rewritten as*

$$S_A(N) = \sum_{u=0}^{N} A_N^u = \lfloor e \cdot N! \rfloor \ (N \geqslant 1), \tag{25}$$

where e is the nature constant and $\Gamma(N+1,1)$ is the "upper" incomplete gamma function written as

$$\Gamma(u+1,x) = \int_x^{\infty} t^u e^{-t} \mathrm{d}t. \tag{26}$$

Proof of Lemma 1. For $S_A(N)$, it can be rewritten as

$$S_A(N) = \sum_{u=0}^{N} A_N^u = N! \left(\sum_{u=0}^{N} \frac{1}{u!} \right). \tag{27}$$

For a non-negative integer N, the 'upper' incomplete gamma function can be rewritten as (Arfken et al., 2011):

$$\Gamma(N+1,x) = \int_x^{\infty} t^N e^{-t} \mathrm{d}t = N! e^{-x} \sum_{u=0}^{N} \frac{x^u}{u!}. \tag{28}$$

Let $x = 1$ in Eq.(28):

$$\Gamma(N+1, x)|_{x=1} = \frac{N!}{e} \sum_{u=0}^{N} \frac{1}{u!}. \tag{29}$$

Combining Eq.(29) and Eq.(27), $S_A(N)$ can be simplified as

$$S_A(N) = \sum_{u=0}^{N} A_N^u = N! \left(\sum_{u=0}^{N} \frac{1}{u!} \right) = e \cdot \Gamma(N+1, 1). \tag{30}$$

Since $\forall N \in \mathbb{Z}^+, u = 0, 1, \ldots, N$, A_N^u is an integer, $S_A(N)$ can be simplified as

$$S_A(N) = e \cdot \Gamma(N+1, 1) = \lfloor e \cdot N! \rfloor, \tag{31}$$

where $\Gamma(N+1, 1) = \frac{\lfloor e \cdot N! \rfloor}{e}$ (Arfken et al., 2011).

□

For convenience, let's define a function:

$$S(N) = \sum_{u=1}^{N} [A_N^u (S_A(u) - 1)]. \tag{32}$$

Then according to Theorem 2, the maximum entropy of RPS can be rewritten as

$$H_{MaxRPS}(N) = \log S(N). \tag{33}$$

Lemma 2 (Approximation of $S(N)$)**.**

$$\begin{aligned} e \cdot N! \left(\sum_{u=1}^{N} \frac{u!}{(N-u)!} - 2 \right) + 1 &\leqslant \lim_{N \to \infty} S(N) \\ &\leqslant e \cdot N! \left(\sum_{u=1}^{N} \frac{u!}{(N-u)!} - 1 \right) + 2. \end{aligned} \tag{34}$$

Proof of Lemma 2. Based on Lemma 1 and Eq.(32), $S(N)$ can be rewritten as

$$\begin{aligned}
S(N) &= \sum_{u=1}^{N}[A_N^u(S_A(u)-1)] \\
&= \sum_{u=1}^{N}[A_N^u(S_A(u))] - \sum_{u=1}^{N} A_N^u \\
&= \sum_{u=1}^{N}[A_N^u(S_A(u))] - \sum_{u=0}^{N} A_N^u + 1 \\
&= \sum_{u=1}^{N}[\frac{N!}{(N-u)!}\lfloor e\cdot u!\rfloor] - \lfloor e\cdot N!\rfloor + 1 \\
&= N!\sum_{u=1}^{N}\frac{\lfloor e\cdot u!\rfloor}{(N-u)!} - \lfloor e\cdot N!\rfloor + 1.
\end{aligned} \tag{35}$$

Suppose that

$$e\cdot u! = \lfloor e\cdot u!\rfloor + \varepsilon_1, \quad e\cdot N! = \lfloor e\cdot N!\rfloor + \varepsilon_2, \tag{36}$$

where $\varepsilon_1, \varepsilon_2 \in [0,1)$.

Then Eq.(35) can be rewritten as

$$\begin{aligned}
S(N) &= N!\sum_{u=1}^{N}\frac{\lfloor e\cdot u!\rfloor}{(N-u)!} - \lfloor e\cdot N!\rfloor + 1 \\
&= N!\sum_{u=1}^{N}\frac{e\cdot u! - \varepsilon_1}{(N-u)!} - (e\cdot N! - \varepsilon_2) + 1 \\
&= e\cdot N!\left(\sum_{u=1}^{N}\frac{u!}{(N-u)!} - 1\right) - \varepsilon_1\cdot N!\sum_{j=0}^{N-1}\frac{1}{j!} + \varepsilon_2 + 1.
\end{aligned} \tag{37}$$

Based on Eq.(27), the $\varepsilon_1\cdot N!\sum\limits_{j=0}^{N-1}\frac{1}{j!}$ in Eq.(37) can be simplified as

$$
\begin{aligned}
\varepsilon_1 \cdot N! \sum_{j=0}^{N-1} \frac{1}{j!} &= \varepsilon_1 \cdot N \cdot (N-1)! \sum_{j=0}^{N-1} \frac{1}{j!} \\
&= \varepsilon_1 \cdot N \cdot e \cdot \Gamma(N,1) \\
&= \varepsilon_1 \cdot e \cdot N! - \varepsilon_1 \varepsilon_3 \cdot N, \qquad (38)
\end{aligned}
$$

where $\varepsilon_3 \in [0,1)$.

Comining Eq.(37) and Eq.(38), $S(N)$ can be rewritten as

$$
\begin{aligned}
S(N) =& e \cdot N! \left(\sum_{u=1}^{N} \frac{u!}{(N-u)!} - 1 \right) \\
&- \varepsilon_1 \cdot e \cdot N! + \varepsilon_1 \varepsilon_3 \cdot N + \varepsilon_2 + 1. \qquad (39)
\end{aligned}
$$

Since $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in [0,1)$, let $\varepsilon_1 \to 1, \varepsilon_2 = \varepsilon_3 = 0$, a lower bound of $S(N)$ is given by

$$
S(N) \geqslant e \cdot N! \left(\sum_{u=1}^{N} \frac{u!}{(N-u)!} - 2 \right) + 1. \qquad (40)
$$

Similarly, let $\varepsilon_1 = 0, \quad \varepsilon_2, \varepsilon_3 \to 1$, $S(N)$ is no greater than the following equation:

$$
S(N) \leqslant e \cdot N! \left(\sum_{u=1}^{N} \frac{u!}{(N-u)!} - 1 \right) + 2. \qquad (41)
$$

Based on Eq.(40) and Eq.(41), Lemma 2 is proved.

$\square$

Theorem 4 (the limit form of the envelope of RPS entropy)**.**

$$\begin{aligned}\lim_{N\to\infty}\ \underset{\mathcal{M}:PES(\Omega)\to[0,1],|\Omega|=N}{C_e(H_{RPSEnt}(\mathcal{M}))} &= \lim_{N\to\infty}\sum_{u=1}^{N}[A_N^u(S_A(u)-1)] \\ &= \lim_{N\to\infty} S(N) = e\cdot(N!)^2. \end{aligned} \tag{42}$$

Proof of Theorem 4. The key aspect of Theorem 4 lies in the part $\lim\limits_{N\to\infty} S(N) = e\cdot(N!)^2$. According to Lemma 2, we take the limit form at both ends of Eq.(34), which is written as

$$\begin{aligned}\lim_{N\to\infty} e\cdot N!\left(\sum_{u=1}^{N}\frac{u!}{(N-u)!}-2\right)+1 &\leqslant \lim_{N\to\infty} S(N) \\ &\leqslant \lim_{N\to\infty} e\cdot N!\left(\sum_{u=1}^{N}\frac{u!}{(N-u)!}-1\right)+2, \end{aligned} \tag{43}$$

as we can see the key component is $\lim_{N\to\infty}\left(\sum_{u=1}^{N}\frac{u!}{(N-u)!}-1\right)$, or $\lim_{N\to\infty}\left(\sum_{u=1}^{N}\frac{u!}{(N-u)!}\right)$ more concisely.

Let $N=2k$, Then $\frac{u!}{(N-u)!}=\frac{1}{1}=1$. Thus, $\sum_{u=1}^{N}\frac{u!}{(N-u)!}\geqslant 1$. We can rewrite this equation as

$$\begin{aligned} 0 &\leqslant \lim_{N\to+\infty}\frac{\sum\limits_{u=1}^{N}\frac{u!}{(N-u)!}}{N!}-1 \\ &= \lim_{N\to+\infty}\sum_{u=1}^{N}\frac{u!}{N!(N-u)!}-1 \\ &= \lim_{N\to+\infty}\left(\sum_{u=1}^{N-3}\frac{u!}{N!(N-u)!}\right)+\left.\frac{u!}{N!(N-u)!}\right|_{u=N-2,N-1,N}-1 \\ &= \lim_{N\to+\infty}\sum_{u=1}^{N-3}\frac{u!}{N!(N-u)!}+\frac{1}{2N(N-1)}+\frac{1}{N}. \end{aligned} \tag{44}$$

Note that $\frac{u!}{N!(N-u)!}$ is a monotonic series that increases as u increases. So $\sum_{u=1}^{N-3} \frac{u!}{N!(N-u)!}$ is no greater than

$$\sum_{u=1}^{N-3} \frac{u!}{N!(N-u)!} \leqslant (N-2) \left.\frac{u!}{N!(N-u)!}\right|_{u=N-3} = \frac{1}{6(N-1)}. \tag{45}$$

Based on Eq.(45), Eq.(44) can be simplified as

$$\begin{aligned} &\lim_{N\to+\infty} \sum_{u=1}^{N-3} \frac{u!}{N!(N-u)!} + \frac{1}{2N(N-1)} + \frac{1}{N} \\ &\quad \leqslant \lim_{N\to+\infty} \frac{7N-3}{6N(N-1)} \\ &\quad = 0. \end{aligned} \tag{46}$$

Therefore, Eq.(44) can be simplified as

$$0 \leqslant \lim_{N\to+\infty} \frac{\sum_{u=1}^{N} \frac{u!}{(N-u)!}}{N!} - 1 \leqslant 0. \tag{47}$$

Thus, the limit form of $\sum_{u=1}^{N} \frac{u!}{(N-u)!}$ is

$$\lim_{N\to\infty} \sum_{u=1}^{N} \frac{u!}{(N-u)!} = N!. \tag{48}$$

Back to Eq.(43), the limit form of both ends is $e \cdot (N!)^2$, namely,

$$\begin{aligned} \lim_{N\to\infty} e \cdot N! \left(\sum_{u=1}^{N} \frac{u!}{(N-u)!} - 2 \right) + 1 &\leqslant \lim_{N\to\infty} S(N) \\ &\leqslant \lim_{N\to\infty} e \cdot N! \left(\sum_{u=1}^{N} \frac{u!}{(N-u)!} - 1 \right) + 2 \end{aligned}$$

$$
\begin{aligned}
e \cdot (N!)^2 \leqslant \lim_{N \to \infty} S(N) \leqslant e \cdot (N!)^2 \\
\Longrightarrow \lim_{N \to \infty} S(N) = e \cdot (N!)^2.
\end{aligned} \tag{49}
$$

Based on Theorem 2, Eq.(14, 32 - 33) and (49), the limit form of the *envelope* of RPS entropy is proved.

$$
\begin{aligned}
\lim_{N \to \infty} \underset{\mathcal{M}:PES(\Omega)\to[0,1],|\Omega|=N}{C_e(H_{RPSEnt}(\mathcal{M}))} &= \lim_{N \to \infty} \exp\left(H_{MaxRPS}\right) \\
&= \lim_{N \to \infty} \sum_{u=1}^{N} \left[A_N^u (S_A(u) - 1)\right] \\
&= \lim_{N \to \infty} S(N) \\
&= e \cdot (N!)^2.
\end{aligned} \tag{50}
$$

Theorem 4 is proved. $\square$

5 Numerical examples and discussion

This section gives some examples to demonstrate the limit discussed earlier. Furthermore, it delves into the relationship between Shannon entropy (Shannon, 1948), Deng entropy (Deng, 2016), and RPS entropy (Chen and Deng, 2023), specifically examining their maximum values.

As the theoretical framework established in previous sections provides powerful insights into the behavior of different entropy measures, we begin with a numerical analysis focusing on large-scale behavior and approximation accuracy, validating these theoretical results and gaining practical understanding of their implications. This approach allows us to examine both the theoretical predictions and their practical computational aspects.

Table 1 Value of $S(N), H_{MaxRPS}$, and their corresponding estimation $S_{lim}(N), H_{LimRPS}$ with different values of N. The error of $S_{lim}(N), H_{LimRPS}$ are marked as $\Delta S, \Delta H$, respectively. While the subscripts in them indicate the relative error or absolute error.

N	$S(N)$	$S_{lim}(N)$	ΔS_{abs}	ΔS_{rel}	H_{MaxRPS}	H_{LimRPS}	ΔH_{abs}	ΔH_{rel}
10	3.96E+13	3.58E+13	-3.79E+12	-9.57E-02	4.52E+01	4.50E+01	-1.45E-01	-3.21E-03
20	1.69E+37	1.61E+37	-8.26E+35	-4.88E-02	1.24E+02	1.24E+02	-7.20E-02	-5.84E-04
30	1.98E+65	1.91E+65	-6.49E+63	-3.28E-02	2.17E+02	2.17E+02	-4.80E-02	-2.22E-04
40	1.85E+96	1.81E+96	-4.58E+94	-2.47E-02	3.20E+02	3.20E+02	-3.60E-02	-1.13E-04
50	2.57E+129	2.51E+129	-5.08E+127	-1.98E-02	4.30E+02	4.30E+02	-2.90E-02	-6.71E-05
60	1.91E+164	1.88E+164	-3.16E+162	-1.65E-02	5.46E+02	5.46E+02	-2.40E-02	-4.41E-05
70	3.96E+200	3.90E+200	-5.61E+198	-1.42E-02	6.66E+02	6.66E+02	-2.10E-02	-3.09E-05
80	1.41E+238	1.39E+238	-1.75E+236	-1.24E-02	7.91E+02	7.91E+02	-1.80E-02	-2.28E-05
90	6.07E+276	6.00E+276	-6.70E+274	-1.11E-02	9.20E+02	9.19E+02	-1.60E-02	-1.74E-05
100	2.39E+316	2.36E+316	-2.38E+314	-9.95E-03	1.05E+03	1.05E+03	-1.40E-02	-1.37E-05

Example 2. *In this example, the comparison between the maximum RPS entropy and the presented limit is on a large scale. Apart from that, the errors between factorial and its approximation–Stirling's formula (Tweddle, 2003),*

$$N! \approx \sqrt{2\pi N}\left(\frac{N}{e}\right)^N \xrightarrow{def} S_t(N), \tag{51}$$

are also presented here to better illustrate the presented limit.

Similar to Stirling's formula which is regarded as an approximation of factorial, the approximation of the *envelope* of RPS entropy is denoted as $S_{lim}(N) = e \cdot (N!)^2$. The values of $S(N)$, H_{MaxRPS}, and their corresponding proposed estimation $S_{lim}(N)$, $H_{LimRPS} = \log S_{lim}(N)$, are listed in Tab. 1. To better illustrate the efficiency and conciseness of the limit form of the *envelope* of RPS entropy, the results of factorial, Stirling's formula, and its absolute errors, relative errors are also listed in Tab. 2.

As shown in Tab. 1 and Tab. 2, as N increases, $S(N)$ and $S_{lim}(N)$ will grow at a rate close to $(N!)^2$, but both of them maintain the same number of digits. Besides, the absolute error between $S_{lim}(N)$ and SN is roughly **one-hundredth** of SN, while the absolute error between $N!$ and Stirling's estimation is approximately **one-thousandth** of $N!$. When taking the logarithmic operation on them, both absolute and relative errors are rapidly reduced to an acceptable range.

Table 2 Value of $N!, \log_2(N!)$ and its corresponding approximation using Stirling's formula. Δ denotes the error, while the texts of subscripts indicate the relative error or absolute error, respectively.

N	$N!$	$S_t(N)$	ΔS_{t-abs}	ΔS_{t-rel}	$\log(N!)$	$\log(S_t(N))$	$\Delta \log_{t-abs}$	$\Delta \log_{t-rel}$
10	3.63E+06	3.60E+06	-3.01E+04	-8.30E-03	2.18E+01	2.18E+01	-1.20E-02	-5.52E-04
20	2.43E+18	2.42E+18	-1.01E+16	-4.20E-03	6.10E+01	6.11E+01	-6.01E-03	-9.84E-05
30	2.65E+32	2.65E+32	-7.36E+29	-2.80E-03	1.08E+02	1.08E+02	-4.01E-03	-3.72E-05
40	8.16E+47	8.14E+47	-1.70E+45	-2.10E-03	1.59E+02	1.59E+02	-3.01E-03	-1.89E-05
50	3.04E+64	3.04E+64	-5.06E+61	-1.70E-03	2.14E+02	2.14E+02	-2.40E-03	-1.12E-05
60	8.32E+81	8.31E+81	-1.15E+79	-1.40E-03	2.72E+02	2.72E+02	-2.00E-03	-7.36E-06
70	1.20E+100	1.20E+100	-1.43E+97	-1.20E-03	3.32E+02	3.32E+02	-1.72E-03	-5.17E-06
80	7.16E+118	7.15E+118	-7.45E+115	-1.00E-03	3.95E+02	3.95E+02	-1.50E-03	-3.81E-06
90	1.49E+138	1.48E+138	-1.37E+135	-9.00E-04	4.59E+02	4.59E+02	-1.34E-03	-2.91E-06
100	9.33E+157	9.33E+157	-7.77E+154	-8.00E-04	5.25E+02	5.25E+02	-1.20E-03	-2.29E-06

Based on the above, it can be concluded that the proposed approximation to maximum RPS entropy is near as good as Stirling's approximation to factorial. When considering the form, the proposed approximation is much more concise compared with Stirling's formula.

Having established the accuracy and efficiency of our approximation for maximum RPS entropy, we now turn our attention to contextualizing these results within the broader framework of entropy measures. By comparing RPS entropy with classical Shannon entropy and Deng entropy, we can better understand its unique characteristics and advantages in uncertainty quantification. The following example provides a systematic comparison of these different entropy measures and their maximum values.

Example 3. *Suppose a finite set with N elements: $\Omega = \{\chi_1, \chi_2, \ldots, \chi_N\}$. Its corresponding sample space, power set, and the PES can be marked as $\Omega, 2^\Omega, PES(\Omega)$, respectively.*

Then $H_{MaxSEnt}$, $H_{MaxDEnt}$, and H_{MaxRPS} can be obtained by the following equations:

$$H_{MaxSEnt} = \log(N), \tag{52}$$

$$H_{MaxDEnt} = \log(3^N - 2^N), \tag{53}$$

$$H_{MaxRPS} = \log(S(N)). \tag{54}$$

In comparison to H_{MaxRPS}, let's define H_{LimRPS} as an approximation to maximum RPS entropy and its relative error~~:~~, ΔH_{RPSEnt}:

$$H_{LimRPS} = \log(e \cdot (N!)^2), \tag{55}$$

$$\Delta H_{RPSEnt} = \frac{H_{LimRPS} - H_{MaxRPS}}{H_{MaxRPS}} \cdot 100\%. \tag{56}$$

When N increases, the different result of $H_{MaxSEnt}, H_{MaxDEnt}, H_{MaxRPS}$ and H_{LimRPS} are shown in Tab. 3 and Fig. 4. And Fig. 5 shows the relative error and absolute error between H_{LimRPS} and H_{RPSEnt}.

As shown in Fig. 4, H_{MaxRPS} and H_{LimRPS} exhibit a greater slope, i.e., a higher growth rate, than $H_{MaxSEnt}$ and $H_{MaxDEnt}$, while $H_{MaxDEnt}$ also exhibit a higher increasing speed in comparison to $H_{MaxSEnt}$. This can be clarified through the simple fact that the uncertainty in a finite set's PES is substantially larger than in its power set or sample space for a given number of elements. This is because RPS considers all permutations of the finite set, while DSET and probability theory do not. DSET, on the other hand, considers all potential subsets of the finite set and may thus be considered an extension of probability theory.

By comparing H_{MaxRPS} and H_{LimRPS} in Tab. 3, Fig. 4 and Fig. 5, it's clear that when $N > 7$, the proposed approximation will converge to H_{MaxRPS} quickly. When $N > 15$, the accuracy of the approximation will reach two decimal errors. Considering the complex computational steps required by the original calculation process, this approximation provides a more reasonable estimation of the maximum RPS entropy. In contrast, the estimation formula of the Stirling formula for the factorial is not concise as the proposed estimation formula for the maximum RPS entropy.

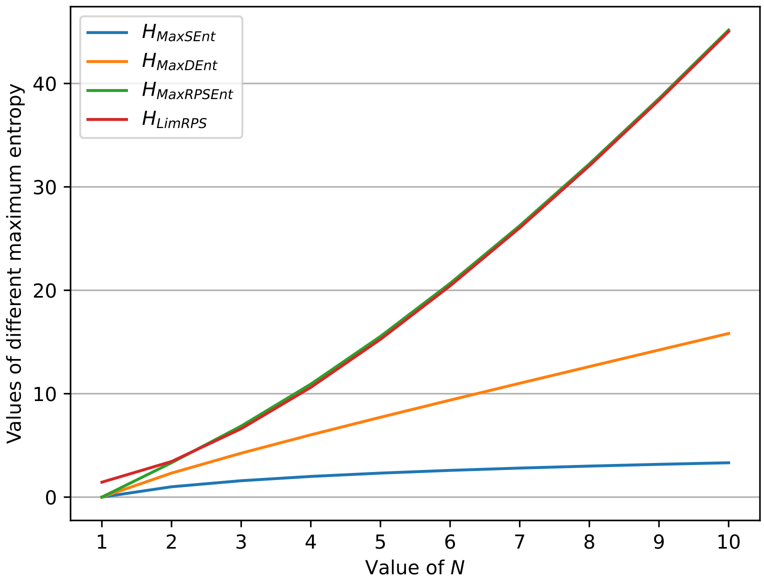

Fig. 4 The trend of maximum Shannon entropy, Deng entropy, RPS entropy and the proposed approximation of maximum RPS entropy when N changes.

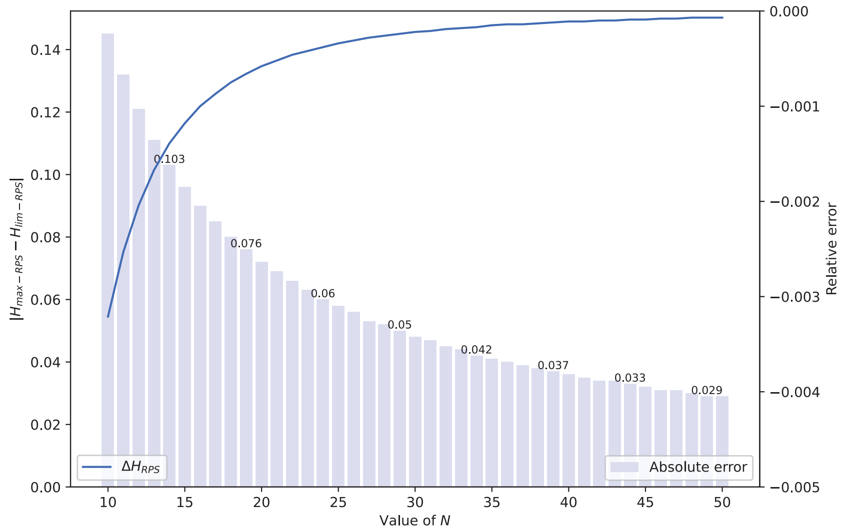

Fig. 5 The trend of relative error and absolute error of the proposed approximation of maximum RPS entropy when N changes. The line chart is denoted as the relative error while the bar chart is denoted as the absolute error.

Table 3 The maximum Shannon entropy, Deng entropy, RPS entropy and the presented result of limit of maximum RPS entropy with different values of N.

N	$H_{MaxSEnt}$	$H_{MaxDEnt}$	H_{MaxRPS}	H_{LimRPS}	ΔH_{RPSEnt}
1	0.00000	0.00000	0.00000	1.44270	0.00%
2	1.00000	2.32193	3.32193	3.44270	3.64%
3	1.58496	4.24793	6.87036	6.61262	-3.75%
4	2.00000	6.02237	10.92780	10.61260	-2.88%
5	2.32193	7.72110	15.54060	15.25650	-1.83%
6	2.58496	9.37721	20.66910	20.42640	-1.17%
7	2.80735	11.00770	26.24950	26.04110	-0.79%
8	3.00000	12.62230	32.22310	32.04110	-0.56%
9	3.16993	14.22660	38.54240	38.38100	-0.42%
10	3.32193	15.82440	45.16990	45.02480	-0.32%

Example 4. *In this example, the computational complexity of the envelope of RPS entropy, as well as the presented limit, is given.*

According to Theorem 3 and Eq.(31), the *envelope* of RPS entropy is

$$\begin{aligned} C_e(H_{RPSEnt}) &= \sum_{a=1}^{N} \left[A_N^a (S_A(a) - 1)\right] \\ &= \sum_{a=1}^{N} \left[\frac{N!}{(N-a)!} \left(\lfloor e \cdot a! \rfloor - 1\right)\right]. \end{aligned} \tag{57}$$

The function performs a summation from $a = 1$ to N, resulting in N iterations and complexity of $O(N)$ or linear time. Within the loop, there are several operations that need to be considered:

- The computation of $N!/(N-a)!$ using the factorial function, which can be done in $O(N)$ time.
- The computation of $\lfloor e \cdot a! \rfloor$ using the factorial and floor functions, also taking $O(a)$ time.
- Subtracting 1, which is a constant time operation $O(1)$.
- Multiplying the results of the above operations, which is also a constant time operation $O(1)$.

Therefore, the operations within the loop have an asymptotic time complexity of $O(N)$. Since there are N iterations, the overall asymptotic complexity becomes $O(N) * O(N) = O(N^2)$.

Regarding the presented limit of the *envelope* of RPS entropy $C_{e(lim)}(H_{RPSEnt} = e \cdot (n!)^2$, its computational complexity is clearly $O(N)$. Considering the accuracy of this approximation, as demonstrated in Example 3 and Example 2, the computational efficiency gained from this approximation will be a significant advantage for future applications. For instance, De Gregorio et al. (2022) proposed an estimation of Shannon entropy to quantify the memory of a given system, and (Irshad et al., 2024) considered an estimation of weighted extropy and used it for reliability modeling.

6 Conclusion

RPS is a great extension of DSET. Though the maximum entropy of RPS is presented, its computational complexity makes it difficult to discuss the *envelope* of RPS entropy. This study addressed this issue by presenting the limit form of the *envelope* of RPS entropy. The result $e \cdot (N!)^2$ establishes a fascinating link between the natural constant e and the factorial function, two fundamental concepts in mathematics.

In future research, there are mainly two problems. The first point of interest is exploring the physical meaning of $e \cdot (N!)^2$ and its potential correlation with RPS. Additionally, while an approximation of the maximum RPS entropy has been proposed, further research is needed to explore its practical applications in specific domains.

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Jiefeng Zhou: Conceptualization, Methodology, Formal analysis, Investigation, Writing-original draft, Writing-review & editing. **Zhen Li**: Validation. **Kang Hao Cheong**: Validation.**Yong Deng**: Writing-review & editing, Supervision, Project administration, Funding acquisition.

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Data Availability

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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